On semisimple ℓ-modular Bernstein-blocks of a p-adic general linear group

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Abstract

Let $G_n = \mathrm{GL}_n(F)$, where F is a non-archimedean local field with residue characteristic p. Our starting point is the Bernstein-decomposition of the representation category of G_n over an algebraically closed field of characteristic $\ell \neq p$ into blocks. In level zero, we associate to each block a replacement for the Iwahori-Hecke algebra which provides a Morita-equivalence just as in the complex case. Additionally, we will explain how this gives rise to a description of an arbitrary G_n-block in terms of simple G_m-blocks (for $m \leq n$), paralleling the approach of Bushnell and Kutzko in the complex setting.

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1 Introduction

Consider the group $G = G_n = \mathrm{GL}_n(F)$, where F is a local non-archimedean field with residue characteristic p. As part of the Bernstein-decomposition, it is now a classical result that the category $\mathfrak{R}_{\mathbb{C}}(G)$ of smooth, complex G-representations decomposes as

$$\mathfrak{R}_{\mathbb{C}}(G) = \Big(\bigoplus_{(\mathscr{P},\rho)} \mathfrak{R}_{\mathbb{C},(\mathscr{P},\rho)}(G) \Big) \ \oplus \ \text{positive-level part}, \qquad (1.1)$$

where

- $(\mathscr{P}, \rho)$ runs through all equivalence classes of level-0 G-types: $\mathscr{P} \subset G$ is a parahoric subgroup (see Definition 2.6) and ρ a $\mathscr{P}$-representation inflated from a supercuspidal representation $\overline{\rho}$ of the reductive quotient $\mathscr{P}/\mathscr{P}(1) = \mathcal{M}$. Up to equivalence (see Definition 2.10), we can assume that $(\mathscr{P}, \rho)$ is in an arranged form, i. e.

$$\mathcal{M} = \prod_{i=1}^{k} \big(\mathrm{GL}_{n_i}(q) \big)^{m_i} \text{ and } \overline{\rho} = \boxtimes_{i=1}^{k} (\overline{\rho}_i)^{m_i},$$

  where $k, n_i, m_i \in \mathbb{N}$ with $\sum_{i=1}^{k} n_i m_i = n$ and each $\overline{\rho}_i$ is a supercuspidal $\mathrm{GL}_{n_i}(q)$-representation with $\overline{\rho}_i \not\cong \overline{\rho}_j$ for $i \neq j$.

- The subcategory $\mathfrak{R}_{\mathbb{C},(\mathscr{P},\rho)}(G)$ consists of representations which have all their irreducible subquotients isomorphic to subquotients of $\mathrm{ind}_{\mathscr{P}}^{G}(\rho)$.

This reduces the representation theory of G to an investigation of the blocks $\mathfrak{R}_{\mathbb{C},(\mathscr{P},\rho)}$. Bushnell and Kutzko provided in [8] a Morita-equivalence[1]

$$\mathfrak{R}_{\mathbb{C},(\mathscr{P},\rho)} \cong \mathscr{H}(G, \mathscr{P}, \rho) - \mathbf{Mod}, \qquad (1.2)$$

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[1]Technically speaking, this is not a Morita-equivalence as there is just one ring. Anyways, the alternative characterisation of (1.2) as $\mathscr{H}(G)_{(\mathscr{P},\rho)} \underset{\text{Morita}}{\cong} \mathscr{H}(G, \mathscr{P}, \rho)$ (with $\mathscr{H}(G)_{(\mathscr{P},\rho)}$ the part of the global Hecke algebra $\mathscr{H}(G)$ lying in $\mathfrak{R}_{\mathbb{C},(\mathscr{P},\rho)}$) justifies this abuse of notation.

where $\mathscr{H}(G, \mathscr{P}, \rho) = \mathrm{End}_G\big(\mathrm{ind}^G_{\mathscr{P}}(\rho)\big)$ is the Iwahori-Hecke algebra associated to the type $(\mathscr{P}, \rho)$. For a simple type (i. e. with $k = 1$), they established in [7] an isomorphism

$$\mathscr{H}(G, \mathscr{P}, \rho) \cong \mathscr{H}(\mathrm{GL}_{m_1}(F^{n_1}), I, \mathbb{C}_{\mathrm{triv}}), \tag{1.3}$$

where F^{n_1} is the unramified extension of F with degree n_1 and I denotes the Iwahori-subgroup of $\mathrm{GL}_{m_1}(F^{n_1})$. Modules over Hecke algebras of this kind were classified by Kazhdan-Lusztig and Ginzburg.

As a final step, Bushnell and Kutzko decompose in [9] the Hecke algebra of a general type as

$$\mathscr{H}(G, \mathscr{P}, \rho) \cong \bigotimes_I \mathscr{H}(G_{m_i n_i}, \mathscr{P}_i, \rho_i), \tag{1.4}$$

where all occurring pairs $(\mathscr{P}_i, \rho_i)$ are simple $G_{m_i n_i}$-types and can henceforth be treated as in (1.3).

If one replaces the base field $\mathbb{C}$ by some algebraically closed field R of positive characteristic $\ell \neq p$, the decomposition (1.1) carries over. Although types and Hecke algebras can still be defined and continue to be an important concept, one loses the Morita-equivalence of (1.2). Therefore, a different concept is needed if one is interested in the structure of the blocks. The main achievement of this paper is the construction of a pair $(\mathscr{P}_{\mathtt{max}}, \tilde{\rho})$ (called the *supercover*) which is a suitable replacement for the type in the sense that it provides both a Morita-equivalence like (1.2) and a tensor-decomposition like (1.4):

Let $(\mathscr{P}, \rho)$ be a type given in the arranged form as described above, then we can form the unique standard parahoric subgroup $\mathscr{P}_{\mathtt{max}}$ with reductive quotient

$$\mathcal{M}_{\mathtt{max}} = \prod_{i=1}^{k} \mathrm{GL}_{n_i m_i}(q).$$

$\mathcal{M}$ is a Levi-subgroup of $\mathcal{M}_{\mathtt{max}}$ and we can consider the Harish-Chandra induced $\mathbf{i}^{\mathcal{M}_{\mathtt{max}}}_{\mathcal{M}}(\overline{\rho})$. Denote by Ψ the set of isomorphism classes of irreducible subquotients of $\mathbf{i}^{\mathcal{M}_{\mathtt{max}}}_{\mathcal{M}}(\overline{\rho})$. Any (representative of an) element of Ψ can be written as $\boxtimes_{i=1}^{k} X_i$, where X_i is an irreducible representation of $\mathrm{GL}_{n_i m_i}(q)$. Any such X_i admits a projective cover $\hat{X}_i$. Then $\hat{X} := \boxtimes_{i=1}^{k} \hat{X}_i$ can be inflated to a $\mathscr{P}_{\mathtt{max}}$-representation $\tilde{X}$ and we set (cf. Definition 3.1)

$$\tilde{\rho} = \bigoplus_{X \in \Psi} \tilde{X}.$$

In Section 4, we study the induced supercover $\mathrm{ind}^G_{\mathscr{P}_{\mathtt{max}}}(\tilde{\rho})$. It follows from certain properties of the Harish-Chandra functor $\mathbf{i}^{\mathcal{M}_{\mathtt{max}}}_{\mathcal{M}}$ (collected in Section 2.6) and a special instance of the Mackey-decomposition for parahoric functors (Section 2.4) that $\mathrm{ind}^G_{\mathscr{P}_{\mathtt{max}}}(\tilde{\rho})$ is a progenerator in $\mathfrak{R}_{R,(\mathscr{P}, \rho)}$. This implies directly the announced Morita-equivalence (Theorem 4.2):

$$\mathfrak{R}_{(\mathscr{P}, \rho)} \cong \mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho}) - \mathbf{Mod}$$

The decomposition of $\mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho})$ as a tensor product is technically more involved. The crucial ingredient is a disjointness argument (Proposition 5.2) based on the work of Dipper, James and Green on (modular) representations of $\mathrm{GL}_n(q)$. In Section 5, we use this to first give an upper bound on the intertwining of

$$\mathrm{Hom}_G\big(\mathrm{ind}_{\mathscr{P}_{\mathtt{max}}}^G(\tilde{X}), \mathrm{ind}_{\mathscr{P}_{\mathtt{max}}}^G(\tilde{Y})\big)$$

for arbitrary $X, Y \in \Psi$. Using the general methods of Section 5.4, we are able to lift this to an upper bound on the intertwining of $\mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho})$ (see Theorem 5.7). This is sufficient to use an argument of Vignéras implying the tensor-decomposition: Denote by $(\mathscr{K}_i, \tilde{\rho}_i)$ the supercover of the simple $\mathrm{GL}_{n_i m_i}(F)$-type $(\mathscr{P}_i, \rho_i)$, where

- $\mathscr{K}_i = \mathrm{GL}_{n_i m_i}(\mathcal{O})$;

- $\mathscr{P}_i$ is the unique parahoric subgroup of $\mathrm{GL}_{n_i m_i}(F)$ with reductive quotient $\mathcal{M}_i = \big(\mathrm{GL}_{n_i}(q)\big)^{m_i}$;

- ρ_i is inflated from the $\mathcal{M}_i$-representation $\overline{\rho}_i^{m_i}$.

Then we establish (cf. Theorem 6.1)

$$\mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho}) \underset{\mathrm{Morita}}{\cong} \bigotimes_I \mathscr{H}(\mathrm{GL}_{n_i m_i}(F), \mathscr{K}_i, \tilde{\rho}_i).$$

We repeat that

$$\mathfrak{R}_{(\mathscr{P}, \rho)}(G) \cong \mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho}) - \mathbf{Mod},$$
$$\mathfrak{R}_{(\mathscr{P}_i, \rho_i)}\big(\mathrm{GL}_{n_i m_i}(F)\big) \cong \mathscr{H}(\mathrm{GL}_{n_i m_i}(F), \mathscr{K}_i, \tilde{\rho}_i) - \mathbf{Mod}.$$

This reduces the study of a general (called *semisimple*) block to simple blocks just as Bushnell-Kutzko theory does in the complex setting. Expressed in sloppy words, this tells us that semisimple blocks are built up from simple ones in the easiest possible way, i. e. all 'mysterious' things happen in the formation of simple blocks from their supercuspidal parts. This is reflected by the fact that cuspidal non-supercuspidal representations (whose existence is a unique feature of the modular case) can occur only in simple blocks[2]. Moreover, from the definition of the supercover it is clear that all modular complications in the representation theory of G come from modular complications in the representation theory of finite linear groups: If for two choices ℓ, ℓ' the representation theories of $\mathrm{GL}_m(q)$ are identical (and we ask for this to hold for all $m \leq n$), then the level-0 parts of the representation theories of G are identical over ℓ and over ℓ'.

In Section 7, we demonstrate how our technique can be used to study the smallest non-trivial example:

- $G = \mathrm{GL}_2(\mathbb{Q}_p)$;

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[2]This follows from a (possibly iterated) application of [19], Claim III.3.15.2.

- (I, ρ) with I the Iwahori subgroup and ρ inflated from a $\mathbb{F}_p^\times \times \mathbb{F}_p^\times$-character $\overline{\rho}_1 \otimes \overline{\rho}_2$ with $\overline{\rho}_1 \not\cong \overline{\rho}_2$.

There is a subgroup $(\mathbb{Z}_p^\times)^{(\ell)}$ of $\mathbb{Z}_p^\times$ with pro-order prime to ℓ such that the quotient is a finite ℓ-group. We can define the subgroup

$$I^{(\ell)} = \begin{pmatrix} (\mathbb{Z}_p^\times)^{(\ell)} & \mathbb{Z}_p \\ \mathfrak{P} & (\mathbb{Z}_p^\times)^{(\ell)} \end{pmatrix} \subset I$$

and prove

$$\mathfrak{R}_{(I,\rho)}(G) \cong \mathscr{H}(G, I^{(\ell)}, \rho|I^{(\ell)}) - \mathbf{Mod}.$$

Additionally, we get

$$\mathscr{H}(G, I^{(\ell)}, \rho|I^{(\ell)}) \cong \bigotimes_{\text{two copies}} R\left[\mathbb{Q}_p/(\mathbb{Z}_p^\times)^{(\ell)}\right],$$

where the category of modules over $R\left[\mathbb{Q}_p/(\mathbb{Z}_p^\times)^{(\ell)}\right]$ is equivalent to the unipotent block (i. e. the block containing the trivial representation) of $\mathfrak{R}(\mathrm{GL}_1(F))$.

It is desireable to generalise results about $\mathrm{GL}_n(F)$ to arbitrary reductive p-adic groups. In our case, the first obstacle would be the Bernstein-decomposition which gets more complicated (cf. Thm. III.6 in [20]). Although it might still be possible to define the supercover in some more general situations, it is ultimately the use of certain results on $\mathrm{GL}_n(q)$ (which are not available in greater generality) what limits our techniques to the situation studied in this paper. This doesn't come as a surprise, given that in the complex case the Bushnell-Kutzko results can be generalised neither easily nor completely to reductive groups. Another question is whether there are connections between the supercover of a simple type and the supercover of (I, R_{triv}) for some other group, paralleling (1.3) in the complex case. Together with the task of generalising the presented results to positive-level blocks, this poses interesting topics for future research.

2 Preliminaries

Fix two prime numbers $p \neq \ell$. Let F be a local non-archimedean field with ring of integers $\mathcal{O}$, some fixed uniformiser ϖ, maximal ideal $\mathfrak{P} = \varpi\mathcal{O}$ and residue field $k \cong \mathbb{F}_q$, where q is some power of p. Moreover, let R be an algebraically closed field of characteristic ℓ such that R arises as residue field in some ℓ-modular system $(R, \mathcal{O}_K, K)$.

The group $G_n = \mathrm{GL}_n(F)$ inherits a topology from F and provides an example of what one calls a locally profinite group (cf. [19]).

Definition 2.1 (Smooth representation)**.** An R-valued representation (V, π) of a locally profinite group G is called smooth if we can find for any $v \in V$ an open subgroup $K \subset G$ which acts trivially on v. Together with G-equivariant linear maps, smooth G-representations define a category denoted by $\mathfrak{R}(G)$.

Depending on the context, we will refer to a representation (V, π) by simply writing V or π. From now on, we will only be concerned with R-valued smooth representations. It is a basic observation (cf. [19]) that $\mathfrak{R}(G)$ is equivalent to the category of modules over the global Hecke algebra

$$\mathscr{H}(G) = \{f : G \to R \,|\, f \text{ locally constant and compactly supported}\}$$

where multiplication is defined as convolution with respect to some chosen Haar measure on G (see [19], I.3.1). This parallels the interpretation of representations of a finite group as modules over the group algebra.

Definition 2.2 (Induction with compact support)**.** Let $H \subset G$ be a closed subgroup of a locally profinite group and consider an H-representation (V, π). Define $\mathrm{ind}_H^G(\pi)_*$ as the space

$$\{f : G \to V \,|\, f(hg) = \pi(h)f(g) \,\forall h \in H, g \in G, f \text{ compactly supported mod-}H\}.$$

The G-action $f \overset{g}{\longmapsto} f(\_g)$ allows us to consider $\mathrm{ind}_H^G(\pi)_*$ as a (possibly not smooth) G-representation. Then define $\mathrm{ind}_H^G(\pi)$ as the biggest subrepresentation of $\mathrm{ind}_H^G(\pi)_*$ which is smooth. (This last step is trivial if H happens to be open.)

There is an obvious way how ind_H^G acts on arrows, allowing us to view induction with compact support as a functor $\mathfrak{R}(H) \to \mathfrak{R}(G)$. If H is open, this functor corresponds (by [19], I.5.2) to

$$\mathscr{H}(H)-\mathbf{Mod} \longrightarrow \mathscr{H}(G)-\mathbf{Mod} \qquad M \mapsto \mathscr{H}(G) \otimes_{\mathscr{H}(H)} M. \qquad (2.5)$$

Let us collect the basic properties:

Proposition 2.1. *Let H be a closed subgroup of G and denote the restriction of a G-representation to an H-representation by res_H^G. Then*

i) *Both res_H^G and ind_H^G commute with direct sums;*

ii) *We have the following adjointness properties: Let $V \in \mathfrak{R}(G)$ and $W \in \mathfrak{R}(H)$, then*

$$\mathrm{Hom}_G\big(\mathrm{ind}_H^G(W), V\big) \cong \mathrm{Hom}_H\big(W, \mathrm{res}_H^G(V)\big) \text{ if } H \text{ is open in } G$$

and

$$\mathrm{Hom}_G\big(V, \mathrm{ind}_H^G(W)\big) \cong \mathrm{Hom}_H\big(\mathrm{res}_H^G(V), W\big) \text{ if } H \text{ is co-compact in } G;$$

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iii) If H is open in G, the functor ind_H^G respects the properties 'cyclic', 'finitely generated' and 'projective';

iv) Let both H_1, H_2 be open subgroups, then we have a Mackey-decomposition

$$\mathrm{res}_{H_2}^G \circ \mathrm{ind}_{H_1}^G \cong \bigoplus_{g \in H_1 \backslash G / H_2} \mathrm{ind}_{H_2 \cap H_1^{g^{-1}}}^{H_2} \circ \mathrm{Int}(g) \circ \mathrm{res}_{H_1 \cap H_2^g}^{H_1};$$

v) The functor res_H^G is exact; Assume there exists an open compact subgroup $K^* \subset G$ whose pro-order is different from zero in R. Then ind_H^G is exact as well.

Proof. The only claim not literally taken from Chapter I.5 of [19] is the 'finitely generated'-part of iii), but this is clear from the characterisation (2.5). $\quad\square$

Remark 2.1. In all cases we are interested in (i. e. G a linear algebraic p-adic group and $\ell \neq p$), the assumption of part v) is fulfilled, see [16], Lemma 1.1.

2.1   Representations of direct products

In this section we will deal with the connections between representations of G, G' and $G \times G'$, where G, G' are two groups. The basic tool will be

Definition 2.3 (Outer tensor product)**.** Let V be a representation of G and V' be a representation of G'. The group $G \times G'$ acts on the space $V \otimes_R V'$ by linear continuation of the rule

$$(g, g')\, v \otimes v' \coloneqq gv \otimes g'v'.$$

The resulting representation of $G \times G'$ is called the outer tensor product and referred to by the symbol $V \boxtimes V'$.

In the two following propositions, by 'linear group' we mean a direct product of finitely many general linear groups.

Proposition 2.2. *The following gives a description of the outer tensor product which generalises to finitely many groups and tensor factors:*

i) Let G, G' be finite groups, then there is an isomorphism of algebras

$$R[G] \otimes_R R[G'] \cong R[G \times G']$$

and under this characterisation $V \boxtimes V'$ corresponds to the space $V \otimes V'$ on which $R[G] \otimes_R R[G']$ acts by linear continuation of the rule

$$(f \otimes f') \cdot (v \otimes v') \coloneqq (f \cdot v) \otimes (f' \cdot v').$$

ii) *Let G, G' be two linear p-adic groups and V a (smooth) G-representation and V' a (smooth) G'-representation. Then $V \boxtimes V'$ is smooth and there is an isomorphism of algebras*

$$\mathscr{H}(G) \otimes_R \mathscr{H}(G') \cong \mathscr{H}(G \times G').$$

Under this characterisation, $V \boxtimes V'$ corresponds to the space $V \otimes V'$ on which $\mathscr{H}(G) \otimes_R \mathscr{H}(G')$ acts by linear continuation of the rule

$$(\varphi \otimes \psi) * (v \otimes v') \coloneqq (\varphi * v) \otimes (\psi * v').$$

Proof. For part i) we refer to Chapter 2.6 of [15].

Considering part ii), we first mention that it is straight-forward to check that $V \boxtimes V'$ is smooth. The isomorphism of the Hecke algebras is provided by linear continuation of the rule

$$\varphi \otimes \psi \mapsto \big[(g, g') \mapsto \varphi(g) \cdot \psi(g')\big].$$

This assignment provides a well-defined and bijective map, as shown in [11], Prop 4.5.5. It follows from a standard Fubini-style theorem (see [11], Thm. 1.4.14 and Remark 2.5.11) that this map commutes with the $*$-multiplication. The remaining claim is checked readily using the definitions. $\qquad\square$

We repeat that, when dealing with p-adic groups, we will assume that all representations under consideration are smooth without mentioning this each and every time. Let us collect some formal properties:

Proposition 2.3. *Consider G-representations V, V_i, W and G'-representations V', W', where G and G' are either both finite or both p-adic linear groups. Then:*

i) *The outer tensor product is distributive:*

$$(V \oplus W) \boxtimes V' = V \boxtimes V' \oplus W \boxtimes V'$$

and analogously in the second variable;

ii) *$V \boxtimes V'$ is cyclic if both V and V' are cyclic (and this is also true if we replace 'cyclic' by 'finitely generated' or by 'finitely generated and projective');*

iii) *If we have a sequence of G-modules*

$$\cdots \to V_{j-1} \to V_j \to V_{j+1} \to \cdots$$

which is exact at j, then so is the sequence

$$\cdots \to V_{j-1} \boxtimes V' \to V_j \boxtimes V' \to V_{j+1} \boxtimes V' \to \cdots$$

at j for any V' (as a sequence of $G \times G'$-modules). The same is true if we fix the G-factor and take a sequence of G'-representations.

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iv) Assume that V and V' are finite-length, then define Γ to be the set of all decomposition factors (up to isomorphism) of V and Γ' analogously for V', then $V \boxtimes V'$ is finite-length and every subquotient is isomorphic to $Q \boxtimes Q'$ for suitable $Q \in \Gamma, Q' \in \Gamma'$;

v) Let $f : V \to V'$ and $f' : W \to W'$ be surjective maps, then so is the induced map $f \boxtimes f' : V \boxtimes W \to V' \boxtimes W'$;

vi) Formation of the outer tensor product provides a bijection

$$\mathrm{Irr}_R(G) \times \mathrm{Irr}_R(G') \overset{1:1}{\longleftrightarrow} \mathrm{Irr}_R(G \times G').$$

The obvious analogues of these assertions hold if we take $G, G', G'', \dots$ some finite collection of groups.

Proof. Regarding i), the map given by linear continuation of

$$f : (v, w) \otimes v' \mapsto (v \otimes v', w \otimes v')$$

is known to provide an isomorphism of vector spaces. f is obviously $G \times G'$-equivariant.

From now on, write R_G for $R[G]$ (if G is finite) or $\mathscr{H}(G)$ (if G is p-adic). The 'cyclic'- and 'finitely generated'-claims of ii) will follow immediately from v). So let both V and V' be finitely generated and projective. Then, by Chapter II, Paragraph 2, no. 2, Cor. to Prop. 4 of [3], $V \oplus Q = R_G^m$ and $V' \oplus Q' = R_{G'}^{m'}$, for a G-representation Q, a G'-representation Q' and two numbers $m, m' \in \mathbb{N}$. Then

$$(V \boxtimes V') \oplus \big((V \boxtimes Q') \oplus (V' \boxtimes Q) \oplus (V' \boxtimes Q')\big) = R_G^m \otimes R_{G'}^{m'} = R_{G \times G'}^{mm'},$$

hence $V \boxtimes V'$ is finitely generated and projective.

The proof for iii) is analogous to i): This claim is true for R-vector spaces and the occurring maps are easily seen to be $G \times G'$-equivariant. iv) is a direct consequence of iii). For v), it is straight-forward to construct a pre-image for any element in $V' \otimes W'$. vi) follows by putting together Proposition 2.2 and [6], Theorem 3.4.2 (and [19], II.2.8, in the p-adic case). $\qquad\square$

Lemma 2.4. *In the notation of the preceding proposition, assume that both V and V' are finitely generated and projective. Then there is a ring-homomorphism*

$$\mathrm{End}_G(V) \otimes_R \mathrm{End}_{G'}(V') \cong \mathrm{End}_{G \times G'}(V \boxtimes V').$$

This generalises to the case where we consider a finite collection of groups $G, G', G'', \dots$.

Proof. Using the characterisation of representations as modules over the appropriate group (or Hecke-) algebra, we can conclude the claim from Exercise 5 in Chapter 9.3 of [18] when both V and V' are finitely presented. But this is the case, as finitely generated projective implies finitely presented (the proof of [4], Chapter 1, Paragraph 2, no. 8, Lemma 8.iii works for non-commutative rings). Part ii of Proposition 2.3 provides an iterative way of generalising the claim to arbitrary finite collections. $\qquad\square$

From now on, we restrict ourselves to the p-adic setting:

Proposition 2.5. *Consider two open, compact subgroups $K \subset G$, $K' \subset G'$ and let V be a K-representation and V' be a K'-representation. Then we have*

$$\mathrm{ind}_K^G(V) \boxtimes \mathrm{ind}_{K'}^{G'}(V') \cong \mathrm{ind}_{K \times K'}^{G \times G'}(V \boxtimes V'). \tag{2.6}$$

Proof. We define a map α from the left to the right side by linear continuation of the following rule:

$$f \boxtimes f' \longmapsto \left[(g, g') \mapsto f(g) \boxtimes f'(g') \right]$$

It is obvious that this is a well-defined $G \times G'$-intertwiner.
The right side of (2.6) is linearly spanned by functions supported on a single coset $(Kg, K'g')$. Let φ be such a function, then it is characterized by its value at (g, g'), say, $\varphi(g, g') = \sum_i v_i \boxtimes v_i'$. Now we can define a map from the right side to the left side of (2.6) by linear continuation of the rule sending φ to $\sum_i f_i \boxtimes f_i'$ with all the f_i supported on Kg and the f_i' supported on $K'g'$ and $f_i(g) = v_i, f_i'(g') = v_i'$. It is easy to see that these maps are inverse to each other. $\qquad\square$

Remark 2.2. If both G and G' are general linear groups over F, the categories $\mathfrak{R}(G)$ and $\mathfrak{R}(G')$ are noetherian by Théorème 5.4.1 in [10]. Thus, in this case, the claim of Lemma 2.4 holds if V and V' are just finitely generated. It might be possible to generalise this to finitely many groups and representations.

2.2   Parabolic and parahoric functors

Recall that, in general for the group $\mathrm{GL}_n(K)$ over some field K, a parabolic subgroup is defined to be the stabiliser of a flag in the vector space K^n. If this flag is adapted to the standard basis we call the resulting parabolic subgroup *standard*. A partition λ of n gives rise to a standard parabolic subgroup, for example associated to $\lambda = (1, 1, \ldots, 1)$ we have the standard Borel subgroup of upper-triangular matrices. Any parabolic subgroup is $\mathrm{GL}_n(K)$-conjugate to a standard one. Recall moreover, that each parabolic subgroup P decomposes as $P = M \ltimes U$, where U is the unipotent radical of P and $M \cong \prod_I \mathrm{GL}_{n_i}(K)$ (for suitable numbers n_i with $\sum_I n_i = n$) is the Levi-factor of P.

Definition 2.4 (Parabolic induction and restriction)**.** Let $P = MU \subset G = \mathrm{GL}_n(K)$ be a parabolic subgroup, where $K = F$ or $K = \mathbb{F}_q$. The parabolic induction functor then transforms an M-representation into a G-representation by first inflating trivially along U (what yields a P-representation) and then inducing up to $\mathrm{GL}_n(K)$ (induction with compact support in the p-adic case). The parabolic restriction associates to a $\mathrm{GL}_n(K)$-representation V the space $V_U = V/V(U)$, where

$$V(U) = \langle v - uv \,|\, v \in V, u \in U \rangle$$

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is the space of U-coinvariants. V_U is naturally a representation of M.

In the p-adic case, these functors are called Jacquet functors and we write $i^G_{M\subset P}, i^G_{M\subset P}$. In the finite case, these functors are called Harish-Chandra functors and we write $\mathbf{i}^G_{M\subset P}, \mathbf{r}^G_{M\subset P}$ or simply $\mathbf{i}^G_M, \mathbf{r}^G_M$. This last notation is justified by the Howlett-Lehrer theorem (cf. [12]), which asserts that the isomorphism class of the induced (or restricted) representation depends only on the Levi-factor and not on the particular parabolic subgroup (which is not true in the p-adic setting).

Definition 2.5 ((Super-)cuspidal representation)**.** An irreducible representation V of G is called cuspidal if $r^G_{M\subset P}(V) \neq 0$ (resp. $\mathbf{r}^G_M(V) \neq 0$) implies $M = G$. V is called supercuspidal if its occurrence as a subquotient of some $r^G_{M\subset P}(W)$ (resp. $\mathbf{i}^G_M(W)$) implies $M = G$.

We collect the basic facts:

Proposition 2.6. *Let $G = \mathrm{GL}_n(K)$ be as above.*

i) *Let $P = MU$ be a parabolic subgroup of G and $Q = NV$ be a parabolic subgroup of M. Then NVU is a parabolic subgroup of G and we have*

$$i^G_{N\subset NVU} \cong i^G_{M\subset P} \circ i^M_{N\subset Q} \quad (\text{ resp. } \mathbf{i}^G_N \cong \mathbf{i}^G_M \circ \mathbf{i}^M_N)$$

and

$$r^G_{N\subset NVU} \cong r^M_{N\subset Q} \circ r^G_{M\subset P} \quad (\text{ resp. } \mathbf{r}^G_N \cong \mathbf{r}^M_N \circ \mathbf{r}^G_M);$$

ii) *The Harish-Chandra functors commute with taking the contragredient representation;*

iii) *Let $P = MU$ and $Q = NV$ be two standard parabolic subgroups of G. Then $M \cap Q$ is a parabolic subgroup of M with unipotent radical $M \cap U$. We have the following Mackey-style decomposition:*

$$\mathbf{r}^G_N \circ \mathbf{i}^G_M = \bigoplus_{g\in P\backslash G/Q} \mathbf{i}^N_{N\cap P^{g-1}} \circ \mathrm{Int}(g) \circ \mathbf{r}^M_{M\cap Q^g}.$$

iv) *Let V be a G-representation and W be an M-representation, then we have the following adjointness relations:*

$$\mathrm{Hom}_M(r^G_{M\subset P}(V), W) \cong \mathrm{Hom}_G(V, i^G_{M\subset P}(W)),$$
$$\mathrm{Hom}_M(\mathbf{r}^G_M(V), W) \cong \mathrm{Hom}_G(V, \mathbf{i}^G_M(W));$$

In the finite case, we have moreover

$$\mathrm{Hom}_G(\mathbf{i}^G_M(W), V) \cong \mathrm{Hom}_M(W, \mathbf{r}^G_M(V)).$$

Proof. Everything can be extracted from [19] or [5]. $\qquad\square$

In the sequel, the letters $\mathcal{G}, \mathcal{P}, \mathcal{M}, \ldots$ will always denote finite groups whereas the notation $G, P, M, \ldots$ will be reserved for the p-adic case. For example, we will use the short-hand notation G_n for $\mathrm{GL}_n(F)$ and $\mathcal{G}_n$ for $\mathrm{GL}_n(q)$.

Consider once again the maximal compact subgroup

$$\mathscr{K} = \mathrm{GL}_n(\mathcal{O}) \subset G = G_n.$$

If we denote by $\mathscr{K}(1) = \mathbf{1} + \mathbb{M}_{n \times n}(\mathfrak{P})$ the pro-p-radical, reduction modulo $\mathscr{K}(1)$ gives a group-homomorphism

$$f_\mathscr{K} : \mathscr{K} \twoheadrightarrow \mathcal{G} = \mathcal{G}_n.$$

Definition 2.6 (Parahoric subgroups)**.** Let $\mathcal{P} = \mathcal{M}\mathcal{U}$ be a standard parabolic subgroup of $\mathcal{G}$. Then the pre-image $\mathscr{P} = f_\mathscr{K}^{-1}(\mathcal{P}) \subset G$ is called a standard parahoric subgroup. $\mathscr{P}$ is open and compact with pro-p-radical $\mathscr{P}(1) = f_\mathscr{K}^{-1}(\mathcal{U})$. Taking the quotient modulo $\mathscr{P}(1)$ defines a group-homomorphism

$$f_\mathscr{P} : \mathscr{P} \twoheadrightarrow \mathcal{M}.$$

A parahoric subgroup is a group G-conjugate to a standard parahoric subgroup.

Definition 2.7 (Parahoric induction and restriction)**.** Retain the notation from the above definition. Then the parahoric induction functor maps an $\mathcal{M}$-representation to a G-representation by firstly inflating trivially along $\mathscr{P}(1)$ (what yields a $\mathscr{P}$-representation) and then inducing compactly to G. The parahoric restriction functor sends a G-representation to its space of $\mathscr{P}(1)$-invariants, which is then understood as a representation of $\mathscr{P}/\mathscr{P}(1) \cong \mathcal{M}$. We denote these functors by $i_\mathscr{P}^G, r_\mathscr{P}^G$. If we are only interested in standard parahoric subgroups, we will use the notation $i_\mathcal{M}^G, r_\mathcal{M}^G$, where $\mathcal{M}$ is a standard Levi-subgroup of $\mathcal{G}$ (the dropping of $\mathscr{P}$ in the notation will be justified by Proposition 2.7.ii).

The standard facts are as follows:

Proposition 2.7. *Retain the notation from the above definitions, then*

  i) *Both functors are exact and parahoric induction commutes with finite direct sums and respects the properties 'cyclic', 'finitely generated' and 'projective';*

  ii) *Let $\mathcal{N} \subset \mathcal{M} \subset \mathcal{G}$, where each inclusion is an inclusion of Levi-subgroups. Then*
$$i_\mathcal{N}^G \cong i_\mathcal{M}^G \circ \mathbf{i}_\mathcal{N}^\mathcal{M} \ and \ r_\mathcal{N}^G \cong \mathbf{r}_\mathcal{N}^\mathcal{M} \circ r_\mathcal{M}^G;$$

  iii) *Let $\mathcal{M}, \mathcal{N}$ be two Levi-subgroups of $\mathcal{G}$, associated to parahoric subgroups $\mathscr{P}, \mathscr{Q}$ of G. Then there is a Mackey-decomposition*
$$r_\mathcal{N}^G \circ i_\mathcal{M}^G \cong \bigoplus_{g \in \mathscr{Q} \backslash G / \mathscr{P}} F_{\mathscr{Q}(q)g\mathscr{P}(q)}^G,$$

  where $F_{\mathscr{Q}(q)g\mathscr{P}(q)}^G$ is the functor given by concatenation of

1. Parabolic restriction along $f_{\mathscr{P}}(\mathscr{P} \cap g^{-1}\mathscr{Q}g) \subset \mathcal{M}$, i. e. taking the $f_{\mathscr{P}}(\mathscr{P} \cap g^{-1}\mathscr{Q}(1)g)$-(co)invariants;

2. The functor between representations of the reductive quotients induced by conjugation

$$\text{Int}(g) : \mathscr{P} \cap g^{-1}\mathscr{Q}g \to g\mathscr{P}g^{-1} \cap \mathscr{Q}$$

(cf. [22], 4.1.1.(e)));

3. Parabolic induction along $f_{\mathscr{Q}}(g\mathscr{P}g^{-1} \cap \mathscr{Q})$;

iv) The set $D = S_n \cdot \Lambda \subset G$ with

$$\Lambda = \big\{ \text{diag}(\varpi^{a_1}, \ldots, \varpi^{a_n}) \,\big|\, a_1, \ldots, a_n \in \mathbb{Z} \big\} \cong \mathbb{Z}^n$$

is a set of representatives for $\mathscr{I} \backslash G / \mathscr{I}$ (with $\mathscr{I} = f_{\mathscr{K}}^{-1}(Borel)$ the Iwahori-subgroup). As $\mathscr{I}$ is contained in any standard parahoric subgroup, we can (e. g. for the purpose of the Mackey-decomposition) choose a set of representatives for $\mathscr{Q} \backslash G / \mathscr{P}$ inside D;

v) Let V be a G-representation and W be an $\mathcal{M}$-representation, then we have an adjointness relation

$$\text{Hom}_G(i_{\mathcal{M}}^G(W), V) \cong \text{Hom}_{\mathcal{M}}(W, r_{\mathcal{M}}^G(V)).$$

Proof. The claim about exactness of induction follows from Proposition 2.1.v because inflation along $\mathscr{P}(1)$ evidently preserves exact sequences. As ℓ does not divide the pro-order of $\mathscr{P}(1)$, the same reasoning can be complemented with Section I.4.6 in [19] to prove exactness of the restriction. The next three parts of i) immediately follow from the corresponding facts for compact induction. The last claim about projectives follows from exactness of parahoric restriction together with part v) of this Proposition and [19], I.A.1.
Part ii) is Claim 4.1.3 of [22].
Part iii) is Proposition 6.4 in [21].
Part iv) can be checked back e. g. with Section 2.3.1 of [17].
Part v) is Claim 4.1.2.e in [22]. $\qquad\square$

As a general rule, we will denote representations of a reductive quotient by $\overline{\rho}, \overline{\sigma}, \ldots$ and the inflations to a parahoric subgroup by $\rho, \sigma, \ldots$.

2.3 An application of the Krull-Remak-Schmidt theorem

Lemma 2.8. *Consider two different ways of writing a finite group as a product of general linear groups over the same field $\mathbb{F}_q$:*

$$\mathfrak{G} = \prod_I \text{GL}_{n_i}(q) = \prod_J \text{GL}_{m_j}(q)$$

with finite index sets I and J. Then there is a bijection $t : I \to J$ such that $n_i = m_{t(i)}$.

Proof. We start by taking the derived subgroup:

$$D(\mathfrak{G}) = \prod_I D_{n_i}(q) = \prod_J D_{m_j}(q),$$

where $D_k(q)$ is short for $D(\mathrm{GL}_k(q))$.

Proposition 2.9. $D_k(q)$ *is directly indecomposable (i. e. cannot be written in a non-trivial way as a direct product) for any choice of $k \in \mathbb{N}^{\geq 2}$ and q any prime power.*

Proof of the proposition. For $k = 2, q = 2$, $\mathrm{GL}_2(2)$ is isomorphic to S_3, hence $D_2(2)$ is isomorphic to the cyclic group with three elements which is indecomposable. For all other values for k and q, $D_k(q)$ is isomorphic to $\mathrm{SL}_k(q)$ ([14], Satz B.3).
It is known that $\mathrm{SL}_k(q)$ is quasi-simple (see [1], Section 31 and [14], Satz B.3, Kor. B.7), hence indecomposable ([1], (31.2)) except for $k = 2, q = 2$ (which was excluded) and $k = 3, q = 2$. In the last case, there are (up to isomorphism) four normal subgroups and the occurring orders are $1, 2, 8$ and $24 = \#(\mathrm{SL}_2(3))$. This is clearly not compatible with a decomposition of $\mathrm{SL}_2(3)$ as a direct product. $\diamond$

For $k = 1$, $D_k(q)$ is the trivial group and hence, by definition, not directly indecomposable.

As the next step, we take the centre of $\mathfrak{G}$ and immediately conclude that $\#(I) = \#(J)$. Define I' as the set of all $i \in I$ such that $n_i > 1$, analogously for J'. Then the proposition allows us to apply the Krull-Remak-Schmidt theorem ([13], Thm. 3.8) to

$$D(\mathfrak{G}) = \prod_I D_{n_i}(q) = \prod_{I'} D_{n_i}(q) = \prod_{J'} D_{m_j}(q).$$

Because $D_k(q)$ is not isomorphic to $D_l(q)$ for $k \neq l$, this tells us that there is a bijection $t : I' \to J'$ such that $n_i = m_{t(i)}$ for all $i \in I'$. As $n_i = m_j = 1$ for $i \in I - I', j \in J - J'$, t can be continued to be a bijection $I \to J$ such that $n_i = m_{t(i)}$ for all $i \in I$. As, again, $D_k(q)$ is not isomorphic to $D_l(q)$ for $k \neq l$, this implies the claim. $\square$

2.4 Special instance of the parahoric Mackey-decomposition

Notation as follows:

- $G_m = \mathrm{GL}_m(F)$ and $G = G_n$;

- $\mathcal{G}_m = \mathrm{GL}_m(q)$ and $\mathcal{G} = \mathcal{G}_n$;

- $\mathscr{P}$ is a standard parahoric subgroup of G with the property that

$$\mathcal{M} = \mathscr{P}/\mathscr{P}(1) = \prod_{i=1,\ldots,k} (\mathcal{G}_{n_i})^{m_i}$$

for numbers $n_i, m_i \in \mathbb{N}$ such that $\sum_{i=1}^k n_i m_i = n$;

- $\overline{\rho}$ denotes an $\mathcal{M}$-representation of the form $\boxtimes_{i=1}^k (\overline{\rho}_i)^{m_i}$, where each $\overline{\rho}_i$ is a supercuspidal $\mathcal{G}_{n_i}$-representation and $i \neq j$ implies $\overline{\rho}_i \not\cong \overline{\rho}_j$ (this is trivially true if $n_i \neq n_j$);

- $\mathcal{K} = \mathrm{GL}_n(\mathcal{O}) \subset G$.

Our aim in this section is to compute $r_{\mathcal{K}}^G \circ i_{\mathscr{P}}^G(\overline{\rho})$, using Proposition 2.7.iii, iv. We say that $d \in D$ survives if $F_{\mathcal{K}(q)d\mathscr{P}(q)}^G$ does not vanish on $\overline{\rho}$. As $\overline{\rho}$ is cuspidal, d survives precisely if the the parabolic-restriction-step in the definition of $F_{\mathcal{K}(q)d\mathscr{P}(q)}^G$ is trivial, i. e. if $\mathscr{P} \cap d^{-1}\mathcal{K}(1)d \subset \mathscr{P}(1)$. As $\mathcal{K}(1)$ is unaffected by S_n-conjugation, this is a condition on λ alone. A straight-forward matrix-computation then shows:

Proposition 2.10. $d = s\lambda$ *survives if and only if* λ *is of the form*

$$\lambda(a_1, \ldots, a_m) =$$
$$\mathrm{diag}(\ \underbrace{\varpi^{a_1}, \ldots, \varpi^{a_1}}_{n_1\text{-}times}, \ldots, \underbrace{\varpi^{a_{m_1}}, \ldots, \varpi^{a_{m_1}}}_{n_1\text{-}times}, \underbrace{\varpi^{a_{m_1+1}}, \ldots, \varpi^{a_{m_1+1}}}_{n_2\text{-}times}, \ldots\ )$$

with $m = \sum_{i=1}^k m_i$ *and* $a_j \in \mathbb{Z}$.

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

The obvious next step is to understand the group $f_{\mathcal{K}}(d\mathscr{P}d^{-1} \cap \mathcal{K})$. Firstly, we write $d = s\lambda$ and remark

$$f_{\mathcal{K}}((s\lambda)\mathscr{P}(s\lambda)^{-1} \cap \mathcal{K}) = s[f_{\mathcal{K}}(\lambda\mathscr{P}\lambda^{-1} \cap \mathcal{K})]s^{-1}$$

because $\mathcal{K}$ is stable under S_n-conjugation and $f_{\mathcal{K}}$ commutes with S_n-conjugation. Thus we have to concentrate on the shape of the group $\mathcal{P}_\lambda := f_{\mathcal{K}}(\lambda\mathscr{P}\lambda^{-1} \cap \mathcal{K})$ with $\lambda = \lambda(a_1, \ldots, a_m)$.

Observation 2.1. $\mathcal{P}_\lambda$ is a parabolic subgroup of $\mathcal{G}$ with Levi-factor $\mathcal{M} = \mathscr{P}/\mathscr{P}(1)$. It's structure is determined by the values

$$\mathrm{sign}(i, j) = \mathbf{1}_{[0,\infty]}(a_i - a_j) - \mathbf{1}_{[-\infty,0)}(a_i - a_j)$$

for $i \neq j \in \{1, \ldots, m\}$. (If e. g. $a_i \leq a_{i+1}$ for all i, we get the standard parabolic subgroup $\mathcal{P} = f_{\mathcal{K}}(\mathscr{P})$. If $a_i > a_{i+1}$ for all i, we get the opposite $\overline{\mathcal{P}}$. It is possible to put this into a clumsy formula, but we don't need that.)

Proof. This follows from the definitions and a matrix calculation. $\qquad\qquad$ $\square$

The next thing to understand is the second step in the definition of $F_{\mathcal{K}(q)d\mathscr{P}(q)}^G$:

Proposition 2.11. *Conjugation by* $d = s\lambda$ *from* $\mathscr{P} \cap d^{-1}\mathcal{K}d$ *to* $d\mathscr{P}d^{-1} \cap \mathcal{K}$ *steps down to conjugation by* s *from the Levi-factor* $\mathcal{M}$ *of* $f_{\mathscr{P}}(\mathscr{P} \cap d^{-1}\mathcal{K}d)$ *to*

the Levi-factor $d\mathcal{M}d^{-1} = s\mathcal{M}s^{-1}$ of $f_{\mathscr{K}}(d\mathscr{P}d^{-1} \cap \mathscr{K})$:

$$
\begin{array}{ccc}
\mathscr{P} \cap d^{-1}\mathscr{K}d & \xrightarrow{\;\mathrm{Int}(d)\;} & d\mathscr{P}d^{-1} \cap \mathscr{K} \\[4pt]
{\scriptstyle f_{\mathscr{P}}}\big\downarrow & & \big\downarrow {\scriptstyle f_{\mathscr{K}}} \\[4pt]
\mathcal{M} & & s\mathcal{P}_\lambda s^{-1} \\[4pt]
\big\| & & \big\downarrow {\scriptstyle p := mod\; s\mathcal{U}_\lambda s^{-1}} \\[4pt]
\mathcal{M} & \xrightarrow[\;\mathrm{Int}(s)\;]{} & s\mathcal{M}s^{-1}
\end{array}
$$

commutes ($\mathcal{U}_\lambda$ denotes the unipotent radical of $\mathcal{P}_\lambda$).

Proof. This becomes clear as soon as we write out the factorisation $\mathrm{Int}(d) = \mathrm{Int}(s) \circ \mathrm{Int}(\lambda)$. Then, as $f_{\mathscr{K}}$ commutes with $\mathrm{Int}(s)$, we have

$$
\begin{array}{ccccccc}
\mathscr{P} \cap d^{-1}\mathscr{K}d & = & \mathscr{P} \cap \lambda^{-1}\mathscr{K}\lambda & \xrightarrow{\;\mathrm{Int}(\lambda)\;} & \lambda\mathscr{P}\lambda^{-1} \cap \mathscr{K} & \xrightarrow{\;\mathrm{Int}(s)\;} & s\lambda\mathscr{P}\lambda^{-1}s^{-1} \cap \mathscr{K} \\[4pt]
& & {\scriptstyle f_{\mathscr{P}}}\big\downarrow & & {\scriptstyle f_{\mathscr{K}}}\big\downarrow & & \big\downarrow {\scriptstyle f_{\mathscr{K}}} \\[4pt]
& & \mathcal{M} & & \mathcal{P}_\lambda & & s\mathcal{P}_\lambda s^{-1} \\[4pt]
& & \big\| & & \big\downarrow & & \big\downarrow \\[4pt]
& & \mathcal{M} & \dashrightarrow_{\;q\;} & \mathcal{M} & \xrightarrow[\;\mathrm{Int}(s)\;]{} & s\mathcal{M}s^{-1}
\end{array}
$$

It is elementary to check that q is the identity. $\qquad\qquad\square$

We conclude that every summand in the Mackey-decomposition must be of the form

$$\mathbf{i}^{\mathcal{G}}_{\mathcal{M}^s \subset \mathcal{Q}^s}(\overline{\rho}^s),$$

where $\mathcal{Q}$ is some parabolic subgroup of $\mathcal{G}$ which admits $\mathcal{M}$ as Levi-factor and s an element of $S_n \subset \mathcal{G}$. This is isomorphic to $\mathbf{i}^{\mathcal{G}}_{\mathcal{M} \subset \mathcal{Q}}(\overline{\rho})$, hence by the Howlett-Lehrer result we have

Theorem 2.12.
$$r^{G}_{\mathscr{K}} \circ i^{G}_{\mathscr{P}}(\overline{\rho}) \cong \bigoplus_{finite} \mathbf{i}^{\mathcal{G}}_{\mathcal{M}}(\overline{\rho}).$$

Proof. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

We need another observation:

Lemma 2.13. *Let $\mathscr{P}, \mathscr{Q}$ be standard parahoric subgroups and π be a $\mathscr{P}$-representation inflated from a cuspidal representation $\overline{\pi}$ of $\mathcal{M} = \mathscr{P}/\mathscr{P}(1)$. Assume that one summand $F^{G}_{\mathscr{Q}(q)d\mathscr{P}(q)}$ in the Mackey-decomposition of $r^{G}_{\mathscr{Q}} \circ i^{G}_{\mathscr{P}}(\overline{\pi})$ contains a cuspidal representation, say $\overline{\sigma}$, of $\mathcal{N} = \mathscr{Q}/\mathscr{Q}(1)$. Then $\mathscr{Q} = \mathscr{P}^g$ and $\sigma \cong \pi^g$ for some $g \in G$, in particular we have*

$$i^{G}_{\mathscr{P}}(\pi) \cong i^{G}_{\mathscr{Q}}(\sigma) \quad \text{and} \quad \bigoplus_{I} \mathbf{i}^{\mathcal{G}}_{\mathcal{M}}(\overline{\pi}) \cong \bigoplus_{I'} \mathbf{i}^{\mathcal{G}}_{\mathcal{N}}(\overline{\sigma})$$

for finite index sets I, I'.

Proof. We will prove this using the language of [22], Section 4.1.1 and 4.1.4: In this notation, $\mathscr{P} = P_J$ and $\mathscr{Q} = P_L$ for two finite, proper subsets J, L of a fixed basis Π for the affine simple roots defined by G and the standard torus. Then $F_{\mathscr{Q}(q)d\mathscr{P}(q)}^G$ is denoted by $F_{L,J}^d$, where d is now taken from some distinguished set of representatives $D_{L,J}$ for $W_L \backslash W_{\mathrm{aff}}(G)/W_J \cong P_L \backslash G/P_J$.

The assumptions of the lemma imply that both $\mathbf{r}_{J \cap d^{-1}L}^J$ and $\mathbf{i}_{L \cap dJ}^L$ are trivial, i. e. $J = J \cap d^{-1}L$ and $L = L \cap dJ$. This implies $L = dJ$. As $P_{dJ} = dP_J d^{-1}$ (4.1.1.(c)), this implies that $\mathscr{P}$ and $\mathscr{Q}$ are conjugate by d. The remaining middle term $\mathrm{Int}(d)$ in the definition of $F_{L,J}^d$ is the reduction of the conjugation $\mathrm{Int}(d) : \mathscr{P} \to \mathscr{Q}$ by 4.1.1.(e). As this maps irreducible representations to irreducible representations, we see that $\overline{\sigma}$ is actually all of $F_{L,J}^d(\overline{\pi})$ and hence that σ and π are d-conjugate. The isomorphism of the parahorically induced representations is therefore established. The last claim results from applying the functor $i_{\mathscr{K}}^G$ and Theorem 2.12. $\qquad\square$

Lemma 2.14. *Let $\mathcal{P} = \mathcal{M}\mathcal{U}$ and $\mathcal{Q} = \mathcal{N}\mathcal{V}$ be two standard parabolic subgroups of $\mathcal{G}$ and assume that*

$$\mathrm{Hom}_{\mathcal{G}}\left(\mathbf{i}_{\mathcal{M}}^{\mathcal{G}}(\pi), \mathbf{i}_{\mathcal{N}}^{\mathcal{G}}(\sigma)\right) \neq 0$$

for a cuspidal $\mathcal{M}$-representation π and a cuspidal $\mathcal{N}$-representation σ. Then $\mathcal{M} = \mathcal{N}^s$ and $\pi \cong \sigma^s$ for some $s \in S_n$. We can even replace s by some suitable $t \in S_n$ which acts merely by rearranging the blocks in the following sense: Write

$$\mathcal{M} = \prod_I \mathcal{G}_{n_i} \text{ and } \mathcal{N} = \prod_J \mathcal{G}_{m_j}.$$

Define $\Theta_k^I = \{i \in I | n_i = k\}$ and $\Theta_k^J = \{j \in J | m_j = k\}$ for $k \in \mathbb{N}$. By Lemma 2.8, $\#\Theta_k^I = \#\Theta_k^J$ and we can define $u := \#I = \#J$. Now define the subset $T\left((m_j)_{j \in J}, (n_i)_{i \in I}\right) \subset \mathrm{Bij}_{\mathrm{Sets}}(\{1, \ldots, u\}, \{1, \ldots, u\})$ which fulfill $t(\Theta_k^J) = \Theta_k^I$ for all k. There is a canonical way to embed $\imath : T \hookrightarrow S_n$ such that conjugation with $\imath(t)$ coincides with the map

$$\mathcal{N} \to \mathcal{M} \qquad (x_j)_{j \in J} \mapsto (x_{t(j)})_{j \in J},$$

where each x_j is an element of $\mathcal{G}_{m_j}$.

Proof. The first part follows from Frobenius reciprocity and an application of the Harish-Chandra Mackey-Theorem (Proposition 2.6 iii).

For the second part, fix some $t_0 \in T\left((m_j)_{j \in J}, (n_i)_{i \in I}\right)$. It is clear s can then be written as $s't_0$, where $s' \in S_n \subset \mathcal{G}$ normalises $\mathcal{N}$. It follows that s' normalises the Young subgroup $S_* = \prod_I S_{n_i}$ defined by $\mathcal{M}$. Borevich and Gavron studied the normaliser of S_* in S_n in [2], and following their exposure we can write $s' = s^1 s_0$, where $s^1 \in S_*$ and $s_0 \in T\left((n_i)_{i \in i}, (n_i)_{i \in I}\right)$. It is obvious that

$$T\left((n_i)_{i \in i}, (n_i)_{i \in I}\right) \cdot T\left((m_j)_{j \in J}, (n_i)_{i \in I}\right) \subset T\left((m_j)_{j \in J}, (n_i)_{i \in I}\right),$$

hence $s = s^1 s_0 t_0$ can be replaced – up to an $\mathcal{M}$-isomorphism, which is provided by s^1 in this case – by the rearrangement of blocks $s_0 t_0 \in T\left((m_j)_{j \in J}, (n_i)_{i \in I}\right)$. $\qquad\square$

2.5 Bernstein-decomposition

Definition 2.8 (Level-0 representation)**.** A representation V is called level-0 if all its irreducible subquotients contain a non-zero vector invariant under the maximal open compact subgroup $\mathscr{K} = \mathrm{GL}_n(\mathcal{O}) \subset G$. V is positive level if it has no level-0 subquotients.

It is well-known that we can split the category of G-representations as

$$\mathfrak{R}(G) = \mathfrak{R}^0(G) \oplus \mathfrak{R}^+(G),$$

where $\mathfrak{R}^0(G)$ is the subcategory of level-0 representations and $\mathfrak{R}^+(G)$ is the subcategory of representations of positive level. In the sequel, we will be concerned with level-0 representations alone.

Definition 2.9 ((Super-)cuspidal pair)**.** Let $M \cong \prod_I G_{n_i} \subset G$ be a Levi-subgroup together with an M-representation $\pi = \boxtimes_I \pi_i$. The pair (M, π) is called cuspidal (resp. supercuspidal) if each π_i is cuspidal (resp. supercuspidal). (M, π) is called level-0 if each π_i is level-0. Two pairs (M_1, π_1) and (M_2, π_2) are said to be G-equivalent[3] if there is a $g \in G$ and an unramified M_2-character χ such that $M_1 = M_2^g$ and $\pi_1 = (\chi \otimes \pi_2)^g$. This equivalence relation respects the notions 'cuspidal', 'supercuspidal' and 'level-0'. The generated equivalence-class is denoted by $[M, \pi]_G$. The set of all equivalence classes of level-0 supercuspidal types is called the (level-0 supercuspidal) Bernstein-spectrum $\mathfrak{B}^{0,\mathrm{sc}}(G)$.

Definition 2.10 (Level-0 (super-)type)**.** Let $\mathscr{P}$ be a parahoric subgroup of G with reductive quotient $\mathcal{M} = \mathscr{P}/\mathscr{P}(1) \cong \prod_J \mathcal{G}_{m_j}$ and a $\mathscr{P}$-representation ρ inflated from $\overline{\rho} = \boxtimes_J \overline{\rho}_j$. The pair $(\mathscr{P}, \rho)$ is called a level-0 type (resp. supertype) if each $\overline{\rho}_j$ is cuspidal (resp. supercuspidal). Two such (super-)types $(\mathscr{P}, \rho)$ and $(\mathscr{P}', \rho')$ are said to be G-equivalent if $\mathrm{ind}_{\mathscr{P}}^G(\rho) \cong \mathrm{ind}_{\mathscr{P}'}^G(\rho')$. Let us denote the set of equivalence classes of supertypes by $\mathfrak{S}^0(G)$.

An $[M, \pi]_G \in \mathfrak{B}^{0,\mathrm{sc}}(G)$ gives rise to a subcategory

$$\mathfrak{R}^{[M,\pi]_G}(G) \subset \mathfrak{R}^0(G)$$

where V is an object if and only if we can associate to each subquotient Q of V an $(N, \sigma) \in [M, \pi]_G$ such that Q is isomorphic to a subquotient of $i_{M \subset P}^G(\sigma)$, where P is some parabolic subgroup of G containing M as Levi-component.

On the other hand, we can associate to the equivalence class defined by a supertype $(\mathscr{P}, \rho)$ a subcategory

$$\mathfrak{R}_{(\mathscr{P},\rho)}(G) \subset \mathfrak{R}^0(G)$$

where V is an object if and only all subquotients of V are subquotients of $\mathrm{ind}_{\mathscr{P}}^G(\rho)$.

---

[3]Some authors use the term *inertially equivalent.*

Theorem 2.15.   *i) The level-0 part of $\mathfrak{R}(G)$ decomposes as*

$$\mathfrak{R}^0(G) = \bigoplus_{[M,\pi]_G \in \mathfrak{B}^{0,sc}(G)} \mathfrak{R}^{[M,\pi]_G}(G) = \bigoplus_{(\mathscr{P},\rho) \in \mathfrak{S}^0(G)} \mathfrak{R}_{(\mathscr{P},\rho)}(G);$$

ii) There exists a bijection between $\mathfrak{B}^{0,sc}(G)$ and $\mathfrak{S}^0(G)$ such that the corresponding subcategories are identical and (up to equivalence) the indices I and J from Definitions 2.9 and 2.10 correspond, i. e. there is a bijection $t : I \rightarrow J$ such that $n_i = m_{t(i)}$ and $\pi_i \cong \pi_{i'}$ if and only if $\overline{\rho}_{t(i)} \cong \overline{\rho}_{t(i')}$.

Proof. Everything is extractable from Chapter IV of [20]. ☐

Depending on the structure of the block (or, equivalently, the type) we distinguish three different sorts of blocks:

1. A *supercuspidal* block is a block generated by a supercuspidal pair of the form (G, π). The associated type is of the form $(\mathscr{K}, \rho)$ with ρ inflated from a supercuspidal $\mathcal{G}$-representation. All simple objects in this block are unramified twists of π.

2. A *simple* block is a block generated by a supercuspidal pair (M, π) with $M = (G_a)^b$ (with $ab = n$) and $\pi = \boxtimes_{b \text{ copies}} \pi_0$. The associated type is of the form $(\mathscr{P}, \rho)$ with $\mathscr{P}/\mathscr{P}(1) \cong (\mathcal{G}_a)^b$ and ρ inflated from $\overline{\rho} = \boxtimes_{b \text{ copies}} \overline{\rho}_0$. In this case, we also call the supercuspidal pair simple. The block generated by the pair (T, R_{triv}) with $T \subset G$ the standard torus (or, alternatively, by the type $(\mathscr{I}, R_{\mathrm{triv}})$ with $\mathscr{I}$ the Iwahori subgroup) is called the *unipotent* block of G. It contains the trivial G-representation.

3. An arbitrary block is called *semisimple*. From now on, we always assume that the associated type is given in the arranged form described at the beginning of Section 2.4.

2.6   Surjectivity of Harish-Chandra induction

Within Brauer-theory, one constructs a decomposition of the representation category $\mathfrak{R}(\mathcal{G}_n)$ of the finite group $\mathcal{G}_n$, which can be seen as the finite analogue of Bernstein's decomposition. We collect some implications of this fact from the literature. We retain the notation from the preceding sections, in particular from Section 2.4. Moreover, we define the subgroup

$$\mathcal{M}_{\mathtt{max}} = \prod_{i=1,\ldots,k} \mathcal{G}_{n_i m_i}$$

of $\mathcal{G}$ which contains $\mathcal{M}$ as a Levi-subgroup.

Proposition 2.16. *Let V be an indecomposable representation of $\mathcal{M}_{\mathtt{max}}$ such that $\mathrm{Hom}_{\mathcal{M}_{\mathtt{max}}}(V, Q) \neq 0$ for some subquotient Q of $\mathbf{i}_{\mathcal{M}}^{\mathcal{M}_{\mathtt{max}}}(\overline{\rho})$. Then all irreducible subquotients of V are isomorphic to irreducible subquotients of $\mathbf{i}_{\mathcal{M}}^{\mathcal{M}_{\mathtt{max}}}(\overline{\rho})$.*

Proof. At first, let us remark that the claim does not depend on the nature of $\mathcal{M}_{\max}$, we could take $\mathcal{G}_n$ instead.

It follows from Section 2.4 in [5] that V decomposes as $V = V_0 \oplus V_1$ such that each irreducible subquotient of V_0 is isomorphic to an irreducible subquotient of $\mathbf{i}_{\mathcal{M}}^{\mathcal{M}_{\max}}(\overline{\rho})$ and such that this does not happen for any irreducible subquotient of V_1. As $\operatorname{Hom}_{\mathcal{M}_{\max}}(V, Q)$ is not empty, V_0 is not trivial. It follows that $V_1 = 0$. $\quad\square$

The next observation depends crucially on the structure of $\mathcal{M}_{\max}$:

Proposition 2.17. *The Harish-Chandra induction* $\mathbf{i}_{\mathcal{M}_{\max}}^{\mathcal{G}}$ *gives an equivalence of categories between* $\mathfrak{R}^{[\mathcal{M},\overline{\rho}]_{\mathcal{M}_{\max}}}(\mathcal{M}_{\max})$ *and* $\mathfrak{R}^{[\mathcal{M},\overline{\rho}]_{\mathcal{G}}}(\mathcal{G})$. *In particular, any irreducible subquotient of* $\mathbf{i}_{\mathcal{M}}^{\mathcal{G}}(\overline{\rho})$ *is of the form* $\mathbf{i}_{\mathcal{M}_{\max}}^{\mathcal{G}}(\mathrm{X})$ *for some irreducible subquotient* X *of* $\mathbf{i}_{\mathcal{M}}^{\mathcal{M}_{\max}}(\overline{\rho})$.

Proof. This is a translation of Lemma 2.4d / Thm. 2.4e in [5] into our language.
$\quad\square$

3 Construction of the supercover

As described in [19], Thm. II.2.4, the irreducible subquotients of $\mathbf{i}_{(\mathcal{G}_{n_i})^{m_i}}^{\mathcal{G}_{n_i m_i}}\left((\overline{\rho}_i)^{m_i}\right)$ may be indexed by partitions μ_i of m_i (but it is not clear that different partitions give rise to non-isomorphic subquotients). We denote these subquotients by P_{i,μ_i}, where μ_i runs through some subset Ξ_i of the set of partitions of m_i such that $\mu_i \neq \mu_i'$ implies $P_{i,\mu_i} \not\cong P_{i,\mu_i'}$. Define

$$\Xi = \prod_{i=1,\ldots,k} \Xi_i \quad \text{and} \quad \Psi = \left\{\boxtimes_{i=1}^k P_{i,\mu_i}\right\}_{(\mu_1,\ldots,\mu_k)\in\Xi}.$$

Then Ψ is a set of representatives (with respect to the equivalence relation 'isomorphism') for the irreducible subquotients of $\mathbf{i}_{\mathcal{M}}^{\mathcal{M}_{\max}}(\overline{\rho})$.

Proposition 3.1. *Let* $X = \boxtimes_{i=1}^k P_{i,\mu_i}$ *be an element of* Ψ. *Then each* P_{i,μ_i} *admits a projective cover* $\hat{P}_{i,\mu_i}$ *and* $\hat{X} = \boxtimes_{i=1}^k \hat{P}_{i,\mu_i}$ *has all its subquotients in* Ψ. *We denote* $\hat{\Psi} = \{\hat{X} | X \in \Psi\}$.

Proof. The first claim follows from [19], A.6.b, and the second claim follows from our Proposition 2.16. $\quad\square$

Consider the standard parahoric subgroup $\mathscr{P}_{\max}$ with finite quotient $\mathcal{M}_{\max}$ and denote the inflation of $\hat{X} \in \hat{\Psi}$ to $\mathscr{P}_{\max}$ by $\tilde{X}$ and the inflation of $X \in \Psi$ by $\tilde{X}^*$. Similarly, for any i and any μ_i, denote the inflation of $\hat{P}_{i,\mu_i}$ to $\mathscr{K}_{n_i m_i} = \operatorname{GL}_{n_i m_i}(\mathcal{O})$ by $\tilde{P}_{i,\mu_i}$ and the inflation of P_{i,μ_i} by $\tilde{P}_{i,\mu_i}^*$.

Lemma 3.2. $\tilde{X}$ *and* $\tilde{P}_{i,\mu_i}$ *are finitely generated and projective (in the category of* $\mathscr{P}_{\max}$*-representations and* $\mathscr{K}_{n_i m_i}$*-representations, resp.).*

Proof. By [19], A.6.b, each P_{i,μ_i} is of finite length. As the property 'finitely generated' is passed on to extensions, 'finite length' implies 'finitely generated'. Now we simply have to put together the proof of Proposition 2.7.i and Proposition 2.3.ii. $\qquad\square$

Indeed, both $\tilde{X}$ and $\tilde{P}_{i,\mu_i}$ are easily seen to be cyclic but we don't need this. Now we are able to define the protagonist of this paper:

Definition 3.1 (Supercover)**.** The pair $(\mathscr{P}_{\mathtt{max}}, \tilde{\rho})$ with

$$\tilde{\rho} = \bigoplus_{X \in \Psi} \tilde{X}$$

is called the supercover of $(\mathscr{P}, \rho)$. The $\mathscr{P}_{\mathtt{max}}$-representation

$$\tilde{\rho}^* = \bigoplus_{X \in \Psi} \tilde{X}^*$$

is a quotient of $\tilde{\rho}$.

Similarly, for each $i \in I$ we have a supercover $(\mathscr{K}_{n_i m_i}, \tilde{\rho}_i)$ of $(\mathscr{P}_i, \rho_i)$, where $\mathscr{P}_i$ is the standard parahoric subgroup of $G_{n_i m_i}$ with reductive quotient $(\mathcal{G}_{n_i})^{m_i}$ and $\tilde{\rho}_i$ is defined by summing over the $\tilde{P}_{i,\mu_i}$ with μ_i running through Ξ_i.

Corollary 3.3. *Both $\tilde{\rho}$ and $\tilde{\rho}_i$ are finite length, finitely generated and projective.*

Proof. This follows from Lemma 3.2 and Proposition 2.3.ii-iii. $\qquad\square$

4 The induced supercover as a progenerator

Let $\mathcal{C}$ be some module category.

Definition 4.1 (Progenerator)**.** $c \in \mathrm{ob}(\mathcal{C})$ is called a progenerator if

1. c is projective in $\mathcal{C}$;

2. c is finitely generated;

3. $\mathrm{Hom}_{\mathcal{C}}(c, c') \neq 0$ for any simple $c' \in \mathrm{ob}(\mathcal{C})$.

Theorem 4.1. *If c is a progenerator, then*

$$\mathcal{C} \cong \mathrm{End}_{\mathcal{C}}(c) - \boldsymbol{Mod}.$$

Proof. Theorem 3.3 in [22]. $\qquad\square$

Let (M, π) be the semisimple supercuspidal pair associated to $(\mathscr{P}, \rho)$, given in the arranged form $M = \prod_I (G_{n_i})^{m_i}$ and $\pi = \boxtimes_I (\pi_i)^{m_i}$.

Theorem 4.2.
$$\mathfrak{R}^{[M,\pi]}(G) \cong \mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho}) - \boldsymbol{Mod}$$
with $\mathscr{H}(G, \mathscr{P}_{\mathtt{max}}, \tilde{\rho}) = \mathrm{End}_G\big(\mathrm{ind}_{\mathscr{P}_{\mathtt{max}}}^G(\tilde{\rho})\big)$.

Proof. We will check that $Y = \mathrm{ind}^G_{\mathscr{P}_{\max}}(\tilde{\rho})$ is a progenerator. Firstly, we have to remark that it follows from Proposition 3.1 that Y indeed lies in $\mathfrak{R}^{[M,\rho]}(G)$. Now we can check with the definition:

1. Projectivity of Y follows from Corollary 3.3 together with Proposition 2.1.iii;

2. The same references tell us that Y is finitely generated;

3. Let $V \in \mathrm{ob}(\mathfrak{R}^{[M,\pi]}(G))$ be irreducible, i. e. V appears as a subquotient in $i^G_{\mathscr{P}}(\overline{\rho})$. As V is level-0, it will not vanish upon application of $r^G_{\mathscr{K}}$. We conclude henceforth from Theorem 2.12 that $r^G_{\mathscr{K}}(V)$ contains a non-zero $\mathcal{G}$-representation which is isomorphic to a subquotient of $\mathbf{i}^{\mathcal{G}}_{\mathcal{M}}(\overline{\rho})$. According to Proposition 2.17, this means that there is some $X \in \Psi$ such that

$$\mathrm{Hom}_{\mathcal{G}}\big(\mathbf{i}^{\mathcal{G}}_{\mathcal{M}_{\max}}(X), r^G_{\mathscr{K}}(V)\big) \neq 0.$$

Using Proposition 2.7.v and 2.7.ii, this implies the existence of a non-zero G-map from $i^G_{\mathscr{P}_{\max}}(X)$ to V, consequently also from $\mathrm{ind}^G_{\mathscr{P}_{\max}}(\tilde{\rho}^*)$ to V. This allows us to finish with the desired conclusion

$$\mathrm{Hom}_G\big(\mathrm{ind}^G_{\mathscr{P}_{\max}}(\tilde{X}), V\big) \neq 0. \qquad \square$$

Of course, this implies at the same time

$$\mathfrak{R}^{[M_i, \pi_i^{m_i}]}(G_{n_i m_i}) \cong \mathscr{H}(G_{n_i m_i}, \mathscr{K}_{n_i m_i}, \tilde{\rho}_i) - \mathbf{Mod}$$

for any $i \in \{1, \ldots, k\}$, where

- $M_i = (G_{n_i})^{m_i}$ is a Levi-subgroup of $G_{n_i m_i}$;

- $\mathscr{K}_{n_i m_i} = \mathrm{GL}_{n_i m_i}(\mathcal{O})$;

- $[M_i, \pi_i^{m_i}]$ is the simple supercuspidal pair associated to the supertype $(\mathscr{K}_{n_i m_i}, \tilde{\rho}_i)$.

5 A bound on intertwining

5.1 Cuspidal and supercuspidal support for $\mathcal{G}_m$

Consider an irreducible representation (V, π) of the finite group $\mathcal{G}_m$. We repeat

Definition 5.1 ((Super-)cuspidal support)**.** There exists a cuspidal representation σ of some Levi-subgroup $\mathcal{M} \subset \mathcal{G}$ such that V is a subrepresentation of $\mathbf{i}^{\mathcal{G}}_{\mathcal{M}}(\sigma)$. $(\mathcal{M}, \sigma)$ is unique up to G-conjugation (see [19], II.2.20) and the G-conjugacy class $[\mathcal{M}, \sigma]^G$ is called the cuspidal support $\mathfrak{Cs}(V)$ of π.
Analogously, there is a supercuspidal representation σ' of some Levi-subgroup $\mathcal{M}'$ such that π is a subquotient of $\mathbf{i}^{\mathcal{G}}_{\mathcal{M}'}(\sigma')$. Then $[\mathcal{M}', \sigma']^G$ is called the supercuspidal support $\mathfrak{Ss}(V)$ of π.

Theorem 5.1 (Strong conjugacy theorem)**.** *Let V be irreducible and $(\mathcal{M}, \pi)$, $(\mathcal{M}', \pi') \in \mathfrak{Cs}(V)$. Then*

$$\mathbf{i}_{\mathcal{M}}^{\mathcal{G}}(\sigma) \cong \mathbf{i}_{\mathcal{M}'}^{\mathcal{G}}(\sigma')$$

and π is both a subrepresentation and a quotient of $\mathbf{i}_{\mathcal{M}}^{\mathcal{G}}(\sigma)$.

Proof. The first part follows from the Howlett-Lehrer result together with [19], I.5.4.iv, and the second part follows from the existence of the contravariant duality explained in [5], in particular Corollary 2.2f. $\qquad\square$

For the next proposition we use the following notation: Consider two factorisations $ab = a'b'$ of some number m. This gives rise to two Levi-subgroups $\mathcal{M} = (\mathcal{G}_a)^b$ and $\mathcal{M}' = (\mathcal{G}_{a'})^{b'}$ of $\mathcal{G}_m$. Additionally, fix a supercuspidal $\mathcal{G}_a$-representation δ and a supercuspidal $\mathcal{G}_{a'}$-representation δ'. Let Q be a subquotient of $\mathbf{i}_{\mathcal{M}}^{\mathcal{G}_m}(\delta^b)$ and Q' be a subquotient of $\mathbf{i}_{\mathcal{M}'}^{\mathcal{G}_m}(\delta'^{b'})$. Then $\mathfrak{Cs}(Q)$ contains an element (M, σ) where M is of the form $\prod_I \mathcal{G}_{c_i}$ (with $\sum_I c_i = m$) and σ of the form $\boxtimes_I \sigma_i$ with the σ_i cuspidal. Just in the same manner, $\mathfrak{Cs}(Q')$ contains an element (M', σ') where M' is of the form $\prod_J \mathcal{G}_{d_j}$ (with $\sum_J d_j = m$) and σ' of the form $\boxtimes_J \sigma'_j$ with the σ'_j cuspidal.

Proposition 5.2. *Assume that δ is not isomorphic to δ' (which has to be checked only if the two factorisations are identical). Then, for any choice $i \in I, j \in J$, the representations σ_i and σ'_j are not isomorphic (which, again, is automatic if c_i does not happen to be equal to d_j).*

Proof. We will use the language and theorems of [19], III.2.3-5. First, δ can be written as $\pi(s, a)$, where $s \in \overline{\mathbb{F}}_q$ is of degree a over $\mathbb{F}_q$. Q is then of the form $\pi(\delta, \mu) = \pi(I)$, where μ is a partition of b and I denotes the ℓ-tête $(s, a; \mu, b)$. We use Lemma III.2.3.1 in [19] to associate to I a certain ℓ-pied spécial $J = ((s, a_j; \mu_j, b_j))_j$ such that $\pi(I) = \pi(J)$ (see Thm. III.2.5 in [19]), where it is important that we extract from the proof of Lemma II.2.3.1 that s indeed equals the one used in the definition of I. According to Cor. III.2.5, we can take as representative of $\mathfrak{Cs}(Q)$

$$M = \prod_j (\mathcal{G}_{a_j})^{b_j} \text{ and } \sigma = \boxtimes_j \pi(s, a_j)^{b_j}$$

We can do the same things with Q' and conclude

$$M' = \prod_{j'} (\mathcal{G}_{a'_{j'}})^{b'_{j'}} \text{ and } \sigma' = \boxtimes_{j'} \pi(s', a'_{j'})^{b'_{j'}},$$

where it is critical that s and s' are not associated (i. e. $s \neq \tau(s)$ for all $\tau \in \mathrm{Gal}(\overline{\mathbb{F}}_q/\mathbb{F}_q)$, see section III.2.2 in [19]) because δ and δ' are not isomorphic. This clearly implies that no factor of σ' can be isomorphic to any factor of σ. $\qquad\square$

5.2 Generalities on intertwining

Consider two parahoric subgroups $\mathcal{Q}, \mathcal{P}$ of G, a $\mathcal{Q}$-representation κ (inflated from a representation $\overline{\kappa}$ of $\mathcal{M} = \mathcal{Q}/\mathcal{Q}(1)$) and a $\mathcal{P}$-representation π (inflated from a representation $\overline{\pi}$ of $\mathcal{N} = \mathcal{P}/\mathcal{P}(1)$). Consider the space

$$M_G(\kappa, \pi) = \operatorname{Hom}_G\left(i_{\mathcal{Q}}^G(\overline{\kappa}), i_{\mathcal{P}}^G(\overline{\pi})\right).$$

Using Proposition 2.7.v (and sticking to the notation of [19], Chapter I.8.5), we can write this as the model

$$M_G'(\kappa, \pi) = \operatorname{Hom}_{\mathcal{M}}\left(\overline{\kappa}, \bigoplus_{g \in \mathcal{Q}\backslash G/\mathcal{P}} F_{\mathcal{Q}(q)g\mathcal{P}(q)}^G(\overline{\pi})\right).$$

Another model is the space $M_G''(\kappa, \pi)$ consisting of maps $f : G \to \operatorname{Hom}_R(V_\pi, V_\kappa)$ which fulfill

- $f(pxq) = \pi(p) \circ f(x) \circ \kappa(q)$ for all $p \in \mathcal{P}, x \in G, q \in \mathcal{Q}$;

- f is supported on finitely many double cosets $\mathcal{P}x\mathcal{Q}$;

(where V_π denotes the underlying space of π).

Definition 5.2 (Intertwining). The Intertwining set $\mathcal{I}_G(\kappa, \pi) \subset \mathcal{P}\backslash G/\mathcal{Q}$ is the set of all double cosets $\mathcal{P}x\mathcal{Q}$ for which there exists an $f \in M_G''(\kappa, \pi)$ such that $f(x) \neq 0$.

Observation 5.1. It follows readily from the explanations in I.8.5 in [19] that $\mathcal{P}x\mathcal{Q} \in \mathcal{I}_G(\kappa, \pi)$ if and only if there is a map in $M_G'(\kappa, \pi)$ which has non-zero contribution to the $\mathcal{Q}x^{-1}\mathcal{P}$-th summand. This is the case if and only if

$$\operatorname{Hom}_{\mathcal{Q} \cap x\mathcal{P}x^{-1}}(\kappa, \pi^x) \neq 0.$$

Remark 5.1. Inspired from this observation, we will say also that $\mathcal{Q}x^{-1}\mathcal{P}$ is in the intertwining set. This can lead to confusion only if $\mathcal{P} = \mathcal{Q}$ and $\pi \neq \kappa$, and we will not encounter this situation in the sequel. Remark that we could have a more uniform notation of this if we used a version of Mackey's decomposition which sums over $\mathcal{P}\backslash G/\mathcal{Q}$ instead of $\mathcal{Q}\backslash G/\mathcal{P}$ (and indeed this is preferred by some authors).

5.3 Intertwining of two subquotients

We introduce the Levi-subgroup

$$M_{\mathtt{max}} = \prod_{i=1}^{k} G_{a_i b_i} \subset G.$$

Theorem 5.3. *Let $X = \boxtimes_{i=1}^{k} X_i$ and $Y = \boxtimes_{i=1}^{k} Y_i$ be two elements of Ψ. Then*

$$\mathrm{Hom}_G \left(i^G_{\mathcal{M}_{\max}}(X), i^G_{\mathcal{M}_{\max}}(Y) \right)$$

has intertwining contained in $\mathscr{P}_{\max} M_{\max} \mathscr{P}_{\max}$.

Proof. <u>Step 1</u>: In accordance with Section 5.1, can find two cuspidal representations π_1, π_2 of (standard) Levi-subgroups $\mathcal{M}_1, \mathcal{M}_2 \subset \mathcal{M}_{\max}$ such that there are maps

$$p : \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_1}(\pi_1) \twoheadrightarrow X \quad \text{and} \quad \imath : Y \hookrightarrow \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2).$$

<u>Step 2</u>: Define a map

$$\Theta : \mathrm{Hom}_G \left(i^G_{\mathcal{M}_{\max}}(X), i^G_{\mathcal{M}_{\max}}(Y) \right) \to \mathrm{Hom}_G \left(i^G_{\mathcal{M}_1}(\pi_1), i^G_{\mathcal{M}_2}(\pi_2) \right)$$

by sending φ to $i^G_{\mathcal{M}_{\max}}(\imath) \circ \varphi \circ i^G_{\mathcal{M}_{\max}}(p)$. By translation into the model M'' of Section 5.2, Θ can be understood as the map

$$M''_G(X, Y) \to M''_G \left(\mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_1}(\pi_1), \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2) \right) \text{ defined by } \Theta(f) : g \mapsto \imath \circ f(g) \circ p.$$

Θ respects the support in the sense that $f(g) \neq 0$ implies $\Theta(f)(g) \neq 0$.
<u>Step 3</u>: By transitivity of the parahoric induction, it is clear that

$$M''_G \left(\mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_1}(\pi_1), \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2) \right) \cong M''_G(\pi_1, \pi_2).$$

This identity is compatible with the intertwining set in the following sense:

Proposition 5.4. *Define $\mathscr{P}_i$ to be the standard parahoric subgroup in G with the property $\mathscr{P}_i/\mathscr{P}_i(1) = \mathcal{M}_i$ $(i = 1, 2)$.*
Let $g_0 \in G$ and assume that $\mathscr{P}_1 k g_0 k' \mathscr{P}_2$ is not in the intertwining of $M''(\pi_1, \pi_2)$ for all choices $k, k' \in \mathscr{P}_{\max}$. Then $\mathscr{P}_{\max} g_0 \mathscr{P}_{\max}$ is not in the intertwining of $M''_G \left(\mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_1}(\pi_1), \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2) \right)$.

Proof of the proposition. Assume, that $\mathscr{P}_{\max} g_0 \mathscr{P}_{\max}$ is in the intertwining of $M''_G \left(\mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_1}(\pi_1), \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2) \right)$. Then there is an f in this set for which we can fix a $g \in \mathscr{P}_{\max} g_0 \mathscr{P}_{\max}$ and write $f(g)(\zeta_1) = \zeta_2 + r$ for suitable choices of

- $\gamma_i \in \mathcal{M}_{\max}$ $(i = 1, 2)$;

- $\zeta_i \in \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_i}(\pi_i)$ non-zero with support $\mathscr{P}_i \cdot \gamma_i$ $(i = 1, 2)$;

- $r \in \mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2)$ supported outside $\mathscr{P}_2 \cdot \gamma_2$.

By replacing g by $\tilde{\gamma}_2^{-1} g \tilde{\gamma}_1^{-1}$ (where $\tilde{\gamma}_i$ denotes a lift of γ_i to $\mathscr{P}_{\max}$), we can assume that $\gamma_1 = \gamma_2 = 1$. In order to prove the claim, we have to construct a non-zero map $\varepsilon \in M''_G(\pi_1, \pi_2)$ with support $\mathscr{P}_2 g \mathscr{P}_1$. This is done as follows: We have two maps

$$s : V_{\pi_1} \hookrightarrow V_{\mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_1}(\pi_1)} \qquad t : V_{\mathbf{i}^{\mathcal{M}_{\max}}_{\mathcal{M}_2}(\pi_2)} \twoheadrightarrow V_{\pi_2}$$

(where, as usual, V_π denotes the space underlying π) given by $s(v) = \xi_v$, where ξ_v has support $\mathscr{P}_1$ and maps p to pv. t is defined by sending a ξ to $\xi(1)$. s intertwines with $\mathscr{P}_1$ and t intertwines with $\mathscr{P}_2$. Hence we can define the map $\varepsilon : G \to \operatorname{Hom}_G(\pi_1, \pi_2)$ supported on $\mathscr{P}_2 g \mathscr{P}_1$ and characterised by sending $p_2 g p_1$ to

$$t \circ f(p_2 g p_1) \circ s = t \circ \mathbf{i}_{\mathcal{M}_2}^{\mathcal{M}_{\max}}(\pi_2)(p_2) \circ f(g) \circ \mathbf{i}_{\mathcal{M}_1}^{\mathcal{M}_{\max}}(\pi_1)(p_1) \circ s = \pi_2(p_2) \circ t \circ f(g) \circ s \circ \pi_1(p_1).$$

$$\diamondsuit$$

Step 4: The strategy now is to show that the intertwining of $M_G(\pi_1, \pi_2)$ lies inside $\mathscr{P}_2 M_{\max} \mathscr{P}_1$. As a consequence of the proposition, this will imply the claim. For this, we use the model M' and write

$$\operatorname{Hom}_{\mathcal{M}_1}\left(\pi_1, \bigoplus_{d \in D} F_{\mathscr{P}_1(q) d \mathscr{P}_2(q)}^G(\pi_2)\right).$$

If there is no d such that there is an f in M' with contribution to the dth summand, there is nothing to prove. If there is such a d, we can apply Lemma 2.13 which tells us that

$$\bigoplus_J \mathbf{i}_{\mathcal{M}_1}^{\mathcal{G}}(\pi_1) \cong \bigoplus_{J'} \mathbf{i}_{\mathcal{M}_2}^{\mathcal{G}}(\pi_2)$$

for two finite index sets J, J'. This, in turn, puts us in a position to apply Lemma 2.14 and conclude that – up to $\mathcal{M}_1$-isomorphism – $(\mathcal{M}_1, \pi_1)$ and $(\mathcal{M}_2, \pi_2)$ are conjugated by a simple rearrangement of blocks $t \in S_n \subset \mathcal{G}$ (as in the formulation of Lemma 2.14). Proposition 5.2 then tells us that t must be contained in $\mathcal{M}_{\max}$. We conclude

$$\mathbf{i}_{\mathcal{M}_1}^{\mathcal{M}_{\max}}(\pi_1) \cong \mathbf{i}_{\mathcal{M}_2}^{\mathcal{M}_{\max}}(\pi_2),$$

i. e. Y is a quotient of $\mathbf{i}_{\mathcal{M}_1}^{\mathcal{M}_{\max}}(\pi_1)$ and we actually have to compute the intertwining of $\mathscr{H}(G, \mathscr{P}_1, \pi_1)$ instead of $M_G(\pi_1, \pi_2)$. It is known that this intertwining is contained in $\mathscr{P}_1 M_{\max} \mathscr{P}_1$ (see [20], Section IV.3.2-3). $\qquad\square$

5.4 Bounds for intertwining pass over to extensions

First, we need a general

Lemma 5.5. *In the module-category over some ring $\mathcal{R}$, consider two short exact sequences*

$$0 \to A \xrightarrow{a} B \xrightarrow{b} C \to 0 \qquad and \qquad 0 \to X \xrightarrow{x} Y \xrightarrow{y} Z \to 0.$$
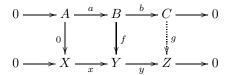
If $\operatorname{Hom}_{\mathcal{R}}(U, V) = 0$ *for all* $U \in \{A, C\}, V \in \{X, Z\}$, *then* $\operatorname{Hom}_{\mathcal{R}}(B, Y) = 0$.

Proof. Assume we have a non-zero $f \in \operatorname{Hom}_{\mathcal{R}}(B, Y)$.
The first observation is that $f \circ a = 0$: Assume, this is not the case, i. e.

there is an $\alpha \in A$ such that $f(a(\alpha)) \neq 0$. Then in any case $y(f(a(\alpha)))$ must vanish, because otherwise we would have produced a non-zero arrow $A \to Z$. So $f(a(\alpha))$ lies in the image of x. This is true for any α' (with $f(a(\alpha'))$ zero or not), hence $f \circ a$ restricts to a map $A \to \mathrm{im}(x) \cong X$. By the assumption on α this map is non-zero and this is a contradiction.

So now we can talk about the following diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{a}{\longrightarrow} & B & \overset{b}{\longrightarrow} & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \vdots{\scriptstyle g} & & \\
0 & \longrightarrow & X & \underset{x}{\longrightarrow} & Y & \underset{y}{\longrightarrow} & Z & \longrightarrow & 0
\end{array}
$$

and the $ABXY$-square is commuting. We will now construct a g making $BCYZ$ commute:

Let $\gamma \in C$. Then we can take a pre-image $b^{-1}(\gamma)$ and consider $f(b^{-1}(\gamma))$. The fact that the left square commutes implies that this element in Y is independent of the choice of the pre-image. Then define $g(\gamma)$ as $y(f(b^{-1}(\gamma)))$. It is straightforward to see that this assignment is $\mathcal{R}$-equivariant. By construction, the $BCYZ$-square commutes. By assumption, $g = 0$. Hence we are in the situation of the following commuting diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \overset{a}{\longrightarrow} & B & \overset{b}{\longrightarrow} & C & \longrightarrow & 0 \\
& & \downarrow{\scriptstyle 0} & & \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle 0} & & \\
0 & \longrightarrow & X & \underset{x}{\longrightarrow} & Y & \underset{y}{\longrightarrow} & Z & \longrightarrow & 0
\end{array}
$$

and we still assume $f \neq 0$.

Return to the assignment $\gamma \mapsto f(b^{-1}(\gamma))$. This indeed is a well-defined $\mathcal{R}$-homomorphism $h : C \to Y$. As f is assumed to be non-zero, so is h. Moreover, $\mathrm{im}(h) = \mathrm{im}(f)$. But $\mathrm{im}(f) \subset \ker(y) = \mathrm{im}(x) \cong X$. Hence we have produced a non-zero map $C \to X$ which gives the final contradiction. $\qquad\square$

Now let M be a (finite-length) module and $\mathcal{Q}$ some set of modules. We say that M decomposes into $\mathcal{Q}$ if there is a sequence of nested submodules

$$M_1 \subset M_2 \subset \ldots \subset M_n = M$$

with M_1 and all the M_i/M_{i-1} for $2 \leq q \leq n$ being isomorphic to members of $\mathcal{Q}$ (and we remark that we do not assume that all members of $\mathcal{Q}$ are irreducible). Define the number $\mathrm{length}_\mathcal{Q}(M)$ to be the smallest $n \in \mathbb{N}$ such that a nested sequence as above exists. The zero-module has $\mathcal{Q}$-length 0 for any $\mathcal{Q}$. If $\mathcal{V}$ is another set of modules, we write $\mathrm{Hom}(\mathcal{Q}, \mathcal{V}) = 0$ if $\mathrm{Hom}(Q, V) = 0$ for any choice $Q \in \mathcal{Q}, V \in \mathcal{V}$.

Corollary 5.6. *Let M, N in $\mathcal{R} - \textbf{Mod}$ be finite-length such that M decomposes into $\mathcal{Q}$ and N into $\mathcal{V}$. Then $\mathrm{Hom}(\mathcal{Q}, \mathcal{V}) = 0$ implies $\mathrm{Hom}(M, N) = 0$.*

Proof. Induction on $d_{M,N} = \max\{\text{length}_{\mathcal{Q}}(M), \text{length}_{\mathcal{V}}(N)\}$: If $d_{M,N} = 0$, the statement is obviously true; So let $d_{M,N} > 1$ and assume the statement is known for all M', N' with $d_{M',N'} < d_{M,N}$. (Additionally, assume that neither M nor N are zero; in that case the claim is true anyway.) Then we can embed M and N into sequences

$$0 \to M' \to M \to Q \to 0 \qquad \text{and} \qquad 0 \to N' \to N \to V \to 0$$

where $Q \in \mathcal{Q}, V \in \mathcal{V}$, M' decomposes into $\mathcal{Q}$, N' decomposes into $\mathcal{V}$ and $d_{M',N'} < d_{M,N}$. The proof now follows from the above lemma. $\square$

We can use this machinery to prove

Theorem 5.7. *The super-Hecke algebra $\mathscr{H}(G, \mathscr{P}_{\max}, \tilde{\rho})$ has intertwining contained in $\mathscr{P}_{\max} M_{\max} \mathscr{P}_{\max}$.*

Proof. Let $g \in G - \mathscr{P}_{\max} M_{\max} \mathscr{P}_{\max}$. The $\mathscr{P}_{\max} \cap g\mathscr{P}_{\max}g^{-1}$-representation $\text{ind}_{\mathscr{P}_{\max}}^G(\tilde{\rho})$ decomposes into $\mathcal{Q} = \{\text{ind}_{\mathscr{P}_{\max}}^G(X) | X \in \Psi\}$ and the $\mathscr{P}_{\max} \cap g\mathscr{P}_{\max}g^{-1}$-representation $\text{ind}_{\mathscr{P}_{\max}}^G(\tilde{\rho})^g$ decomposes into $\mathcal{Q}^g = \{\text{ind}_{\mathscr{P}_{\max}}^G(X)^g | X \in \Psi\}$. By Theorem 5.3,
$$\text{Hom}_{\mathscr{P}_{\max} \cap g\mathscr{P}_{\max}g^{-1}}(\mathcal{Q}, \mathcal{Q}^g) = 0.$$

The claim therefore follows from Corollary 5.6. $\square$

6 Factorisation of the Hecke algebra of a super-cover

In the last section, we showed (Theorem 5.7) that the subspace

$$\mathscr{H}(\mathscr{P}_{\max} M_{\max} \mathscr{P}_{\max}, \mathscr{P}_{\max}, \tilde{\rho}) \subset \mathscr{H}(G, \mathscr{P}_{\max}, \tilde{\rho})$$

of functions with support in $\mathscr{P}_{\max} M_{\max} \mathscr{P}_{\max}$ is actually all of $\mathscr{H}(G, \mathscr{P}_{\max}, \tilde{\rho})$, hence it is an R-algebra. This allows us to use Proposition II.8 and Proposition II.4 of [20], which tells us that there is an isomorphism of algebras

$$\mathscr{H}(G, \mathscr{P}_{\max}, \tilde{\rho}) \cong \mathscr{H}(M_{\max}, \mathscr{P}_{\max}^\circ, \tilde{\rho}^\circ),$$

where $\mathscr{P}_{\max}^\circ = \mathscr{P}_{\max} \cap M_{\max}$ and $\tilde{\rho}^\circ = \tilde{\rho}|\mathscr{P}_{\max}^\circ$. Using this, we can show

Theorem 6.1. *There are numbers $u_i \in \mathbb{N}$ such that*

$$\mathscr{H}(G, \mathscr{P}_{\max}, \tilde{\rho}) \cong \bigotimes_{i \in I} \mathscr{H}\big(G_{n_i m_i}, \mathscr{K}_{n_i m_i}, \oplus_{u_i \, copies} \, \tilde{\rho}_i\big).$$

The i-th tensor factor is Morita-equivalent to

$$\mathscr{H}(G_{n_i m_i}, \mathscr{K}_{n_i m_i}, \tilde{\rho}_i).$$

Proof. Let's unravel the definitions:

- $M_{\mathtt{max}} = \prod_I G_{n_i m_i}$;

- $\mathscr{P}^{\circ}_{\mathtt{max}} = \prod_I \mathscr{K}_{n_i m_i}$;

- $\tilde{\rho}^{\circ} = \boxtimes_I \left(\oplus_{u_i \mathrm{copies}} \tilde{\rho}_i \right)$ with $u_i = \frac{\#\Xi}{\#\Xi_i}$.

The first claim follows now from applying Proposition 2.5 followed by Lemma 2.4. For the second claim, we see that the i-th factor is equal to

$$\mathrm{End}_{G_{n_i m_i}} \left(\oplus_{u_i \mathrm{copies}} \; \mathrm{ind}^{G_{n_i m_i}}_{\mathscr{K}_{n_i m_i}} (\tilde{\rho}_i) \right) \cong \mathbb{M}_{u_i \times u_i} \left(\mathscr{H}(G_{n_i m_i}, \mathscr{K}_{n_i m_i}, \tilde{\rho}_i) \right)$$

and this ring is Morita-equivalent to $\mathscr{H}(G_{n_i m_i}, \mathscr{K}_{n_i m_i}, \tilde{\rho}_i)$. $\qquad\square$

7   A worked example

We conclude with working out the example

- $G = \mathrm{GL}_2(\mathbb{Q}_p)$;

- $M = T$ and $\pi = \pi_1 \boxtimes \pi_2$, where π_i are level-0 characters such that π_1/π_2 is ramified;

- Let χ_i be the restriction of π_i to $\mathbb{Z}_p^{\times}$. Then χ_i is inflated from a character $\overline{\chi}_i$ of $\mathbb{Z}_p^{\times}/(1+\mathfrak{P}) \cong k^{\times}$. The associated type to (M, π) is (I, χ), where I is the Iwahori-subgroup of G and χ is inflated from $\overline{\chi}_1 \boxtimes \overline{\chi}_2$.

Decompose $s = k^{\times}$ as $s = s_{\ell} \times s^{(\ell)}$, where s_{ℓ} is an ℓ-group and the order of $s^{(\ell)}$ is prime to ℓ. We also set $\mathcal{T} = s \times s = \mathcal{T}_{\ell} \times \mathcal{T}^{(\ell)} = (s_{\ell} \times s_{\ell}) \times (s^{(\ell)} \times s^{(\ell)})$. If θ is the projection $\mathbb{Z}_p^{\times} \twoheadrightarrow k^{\times}$, denote by $(\mathbb{Z}_p^{\times})^{(\ell)}$ the pre-image of $s^{(\ell)}$ under θ. This gives rise to a subgroup

$$I^{(\ell)} = \begin{pmatrix} (\mathbb{Z}_p^{\times})^{(\ell)} & \mathbb{Z}_p \\ \mathfrak{P} & (\mathbb{Z}_p^{\times})^{(\ell)} \end{pmatrix}$$

of I. We have

- $I/I(1) = \mathcal{T}$;

- $I^{(\ell)}/I(1) = \mathcal{T}^{(\ell)}$;

- $I/I^{(\ell)} = \mathcal{T}_{\ell}$.

Inflation among $I(1)$ defines two functors

$$\mathrm{infl}^I_{\mathcal{T}} : \mathcal{T}-\mathbf{Mod} \to I-\mathbf{Mod};$$
$$\mathrm{infl}^{I^{(\ell)}}_{\mathcal{T}^{(\ell)}} : \mathcal{T}^{(\ell)}-\mathbf{Mod} \to I^{(\ell)}-\mathbf{Mod}.$$

Proposition 7.1. *The functors* $\mathrm{ind}^I_{I^{(\ell)}} \circ \mathrm{infl}^{I^{(\ell)}}_{\mathcal{T}^{(\ell)}}$ *and* $\mathrm{infl}^I_{\mathcal{T}} \circ \mathrm{ind}^{\mathcal{T}}_{\mathcal{T}^{(\ell)}}$ *are isomorphic.*

Proof. Let (π, V) be in $\mathcal{T}^{(\ell)}-\mathbf{Mod}$. $\mathrm{ind}_{I^{(\ell)}}^{I} \circ \mathrm{infl}_{\mathcal{T}^{(\ell)}}^{I^{(\ell)}}(V)$ consists of all maps

$$f : I \to V \text{ such that } f(i^{(\ell)}i) = \pi(\overline{i^{(\ell)}})f(i) \text{ for all } i^{(\ell)} \in I^{(\ell)}, i \in I,$$

where $\overline{i^{(\ell)}} = \theta(i^{(\ell)})$ and where $i \in I$ acts by $f \mapsto f(\_i)$. $\mathrm{infl}_{\mathcal{T}}^{I} \circ \mathrm{ind}_{\mathcal{T}^{(\ell)}}^{\mathcal{T}}(V)$ on the other hand consists of all maps

$$\varphi : \mathcal{T} \to V \text{ such that } \varphi(t^{(\ell)}t) = \pi(t^{(\ell)})\varphi(t) \text{ for all } t^{(\ell)} \in \mathcal{T}^{(\ell)}, t \in \mathcal{T},$$

where $i \in I$ acts by $\varphi \mapsto \varphi(\_\bar{i})$. It is clear that the assignment $\varphi \mapsto \varphi \circ \theta$ gives rise to the desired isomorphism. □

Proposition 7.2. *Let Γ be a (locally) profinite group, H a normal compact subgroup such that Γ/H is a finite ℓ-group. If χ is an H-character which admits a continuation to Γ, then $\mathrm{ind}_{H}^{\Gamma}(\chi)$ is indecomposable.*

Proof. Denote the continuation by $\tilde{\chi}$, then we have $\mathrm{ind}_{H}^{\Gamma}(\chi) = \tilde{\chi} \otimes \mathrm{ind}_{H}^{\Gamma}(1)$ by [19], I.5.2.d. It is an elementary observation that $\mathrm{ind}_{H}^{\Gamma}(1) \cong R[H \backslash \Gamma]$, where $\gamma \in \Gamma$ acts on $R[H \backslash \Gamma]$ by multiplication with γ^{-1} from the right. By Brauer theory, $R[H \backslash \Gamma]$ is indecomposable. ($H \backslash \Gamma$ is a finite ℓ-group, hence its group-algebra decomposes into indecomposable blocks, and there is a 1-to-1-correspondence between these blocks and ℓ-regular classes. There is only the trivial ℓ-regular class.) □

Lemma 7.3. $\mathrm{ind}_{(\mathbb{Z}_p^\times)^{(\ell)}}^{\mathbb{Z}_p^\times}(\chi_i)$ *is the projective cover of χ_i and $\mathrm{ind}_{\mathcal{T}^{(\ell)}}^{\mathcal{T}}(\overline{\chi}_i)$ is the projective cover of $\overline{\chi}_i$.*

Proof. The induced representation is projective (see also [19], I.4.6, in the $\mathbb{Z}_p^\times$-case) and admits χ_i (resp. $\overline{\chi}_i$) as a quotient. By the above proposition it is also clear that it is indecomposable. This is sufficient to conclude the statement by [19], A.4. □

Now the preceding results yield

$$\mathfrak{R}^{[(M,\pi)]} \cong \mathscr{H}(G, I^{(\ell)}, \chi|I^{(\ell)})-\mathbf{Mod}$$

and

$$\mathscr{H}(G, I^{(\ell)}, \chi|I^{(\ell)}) \cong \mathscr{H}(\mathbb{Q}_p, (\mathbb{Z}_p^\times)^{(\ell)}, \chi_1) \otimes \mathscr{H}(\mathbb{Q}_p, (\mathbb{Z}_p^\times)^{(\ell)}, \chi_2)$$
$$\cong \bigotimes_{\text{two copies}} R\left[\mathbb{Q}_p/(\mathbb{Z}_p^\times)^{(\ell)}\right],$$

where

$$R\left[\mathbb{Q}_p/(\mathbb{Z}_p^\times)^{(\ell)}\right]-\mathbf{Mod} \cong \text{Unipotent block of } \mathfrak{R}\big(\mathrm{GL}_1(F)\big).$$

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